Deep Waters

Research Notes

September 24, 2026

What We Do

- **Build on the hydrologic framework** that tracks local water stocks, flows, recharge, runoff, and depletion.
- **Add an economic structure** with multi-country, multi-grid, multi-sector production, stochastic land productivity, consumption, and trade.
- **Integrate hydrologic dynamics into the economic model through two margins:**
  - **Within period:**
    - Hydrologic states determine local **water availability and mean land efficiency**.
    - Given these hydrologic states, the static general equilibrium determines **production, land allocation, and trade**.
  - **Across time:**
    - Extraction changes future **water stocks and hydrologic states**.
    - A **Hotelling-type condition** characterizes the **intertemporal dynamics of water extraction**.
- **Today:**
  - Present the static economic structure of the model;
  - Preview the intertemporal water-extraction maximization problem.

Spatial Structure and Indices

- **Countries:** $n, m \in \mathcal{N} \equiv \{1, \dots, N\}$.
- **Physical grids:** $i \in \mathcal{I} \equiv \{1, \dots, I\}$, with

  $$\mathcal{I} = \biguplus_{n \in \mathcal{N}} \mathcal{I}_n, \qquad \mathcal{I}_n \cap \mathcal{I}_m = \varnothing \ (n \neq m).$$

  Each grid belongs to exactly one country; i_n denotes grid i located in country n.
- **Sectors:** $j, k \in \mathcal{J} \equiv \{1, \dots, J\}$; $\mathcal{J}^L \subseteq \mathcal{J}$ denotes active land-using sectors.
- **Water sources and time:** $a \in \{\mathrm{Sh}, \mathrm{Dp}\}$ indexes shallow-groundwater and deep-groundwater extraction; $t = 0, \dots, T$.

Appendix: water states and motion

Water Extraction and Local Water Endowment

- In grid i_n, water can be extracted from two sources:

$$a \in \{\mathrm{Sh}, \mathrm{Dp}\}.$$

- Total extraction depth is

$$\mathsf{Div}_{i_n,t} \equiv \mathsf{Div}^{\mathrm{Sh}}_{i_n,t} + \mathsf{Div}^{\mathrm{Dp}}_{i_n,t}.$$

- Let $\mathcal{A}_{i_n}$ denote grid area. The water endowment available to production at time t is

$$\bar{F}^{\mathrm{Water}}_{i_n,t} = \mathcal{A}_{i_n} \mathsf{Div}_{i_n,t}.$$

Economic Model Overview

- **Spatial economy**
  - Multiple countries, multiple grids within each country, and multiple sectors.
- **Sectoral goods production**
  - Combine a CDE primary-factor composite and sectoral composite intermediates.
  - The CDE composite includes labor, local water, and land in efficiency units.
  - Sectoral composite intermediates link sectors through roundabout production.
- **Land**
  - Each grid contains a continuum of plots $\omega \in [0, 1]$.
  - Sector-specific Fréchet draws in land efficiency imply stochastic plot-level returns.
  - Households allocate plots in two layers:
    Active vs. Idle $\Rightarrow$ conditional on Active: across land-using sectors.
- **Households**
  - Receive labor, land, and water income; consume sectoral goods.
- **Trade**
  - Grid intermediate output is collected at national hubs before international trade.
  - Two-layer CES Armington demand: H vs. F $\Rightarrow$ conditional on F : $F_1, \ldots, F_N$.

Model: Production I — CDE Primary Factors and Upper Nest

- The quantity of the primary-factor composite, $V_{i_n,j,t}$, is defined implicitly by a CDE production function over labor, local water, and land in efficiency units:

$$V_{i_n,j,t} = \mathcal{V}^{\text{CDE}}_{n,j}\left(F^{\text{LAB}}_{i_n,j,t}, F^{\text{Water}}_{i_n,j,t}, e_{i_n,j,t}(\omega) F^{\text{Land}}_{i_n,j,t}; \boldsymbol{\beta}_{n,j}, \boldsymbol{\sigma}_j, \boldsymbol{\epsilon}_j\right).$$

- The corresponding CDE unit-expenditure relation is

$$1 = \beta^{\text{LAB}}_{n,j} {V_{i_n,j,t}}^{\epsilon^{\text{LAB}}_j (1-\sigma^{\text{LAB}}_j)} \left(\frac{w^{\text{LAB}}_{n,t}}{P^V_{i_n,j,t} V_{i_n,j,t}}\right)^{1-\sigma^{\text{LAB}}_j} + \beta^{L}_{n,j} {V_{i_n,j,t}}^{\epsilon^{L}_j (1-\sigma^{L}_j)} \left(\frac{r^{\text{Land}}_{i_n,j,t}(\omega)/e_{i_n,j,t}(\omega)}{P^V_{i_n,j,t} V_{i_n,j,t}}\right)^{1-\sigma^{L}_j} + \beta^{W}_{n,j} {V_{i_n,j,t}}^{\epsilon^{W}_j (1-\sigma^{W}_j)} \left(\frac{r^{\text{Water}}_{i_n,t}}{P^V_{i_n,j,t} V_{i_n,j,t}}\right)^{1-\sigma^{W}_j}$$

$P^V_{i_n,j,t}$ is the unit-expenditure price of the primary-factor composite.

- The upper production nest combines the primary-factor composite and sectoral intermediates:

$$Q_{i_n,j,t} = A^{\text{prod}}_{n,j,t} {V_{i_n,j,t}}^{\gamma^V_{n,j}} \prod_{k \in \mathcal{J}} \left(M^{int}_{i_n,j,k,t}\right)^{\gamma_{n,j,k}}, \qquad \gamma^V_{n,j} + \sum_k \gamma_{n,j,k} = 1.$$

Appendix

Model: Production II — Unit Cost and Land Return

- Perfect competition implies the grid–sector unit cost of production:

$$c_{i_n,j,t} = \mathcal{C}_{n,j}\left(w^{\mathrm{LAB}}_{n,t}, r^{\mathrm{Water}}_{i_n,t}, \frac{r^{\mathrm{Land}}_{i_n,j,t}(\omega)}{e_{i_n,j,t}(\omega)}, \{P_{n,k,t}\}_{k\in\mathcal{J}}, V_{i_n,j,t}\right).$$

- Inverting this relation with respect to the land return gives

$$r^{\mathrm{Land}}_{i_n,j,t}(\omega) = e_{i_n,j,t}(\omega)c_{i_n,j,t}\hbar_{i_n,j,t}.$$

  - $\hbar_{i_n,j,t}$ collects the grid–sector components common across plots, isolating $e_{i_n,j,t}(\omega)$ as the plot-specific stochastic component.

- Current hydrologic states determine mean land efficiency:

$$E_{i_n,j,t} \equiv \mathbb{E}[e_{i_n,j,t}(\omega)] = \bar{E}_{i_n,j,t}g_j(M_{i,t}, S_{i,t}, D_{i,t}).$$

  - $\bar{E}_{i_n,j,t}$ captures baseline grid–sector land efficiency, while $g_j(\cdot)$ captures the sector-specific effect of current local hydrologic conditions on land efficiency.

Appendix: land-return inversion

Land Allocation: Land Manager's Problem

- Each physical plot $\omega \in [0,1]$ chooses between an active land use $j \in \mathcal{J}^L$ and remaining idle:

$$\max \left\{ \left\{ r^{\mathrm{Land}}_{i_n,j,t}(\omega) \right\}_{j \in \mathcal{J}^L},\ r^{\mathrm{Land}}_{i_n,0,t}(\omega) \right\}.$$

- **Active land:** $e_{i_n,j,t}(\omega)$ is the plot–sector-specific land-efficiency draw:

$$r^{\mathrm{Land}}_{i_n,j,t}(\omega) = e_{i_n,j,t}(\omega) c_{i_n,j,t} \hbar_{i_n,j,t}, \qquad j \in \mathcal{J}^L.$$

- **Idle land:** $e_{i_n,0}(\omega)$ is the plot-specific replacement/setup requirement, measured in labor units. Its labor cost is

$$r^{\mathrm{Land}}_{i_n,0,t}(\omega) = e_{i_n,0}(\omega) w^{\mathrm{LAB}}_{n,t}.$$

- The land-efficiency and replacement-cost draws follow a two-layer nested Fréchet distribution.

$$\Pr(\mathbf{e}_{i_n,t}(\omega) \le \mathbf{e}_{i_n,t}) = \exp\left\{ -[\Gamma(1-1/\nu_1)]^{-\nu_1} \left[\left(\frac{e_{i_n,0}}{E_{i_n,0}} \right)^{-\nu_1} + \left(\sum_{j \in \mathcal{J}^L} \left(\frac{e_{i_n,j}}{E_{i_n,j,t}} \right)^{-\nu_2} \right)^{\nu_1/\nu_2} \right] \right\}.$$

  - ν_2 governs substitution across active land uses; ν_1 governs the active–idle margin.

Land Allocation: Two-Layer Mapping

- Let $\bar{F}_i^{\text{Land}}$ denote the fixed physical-land endowment in grid i.
- The upper margin determines the active-land share $\pi^L_{i_n,A,t}$; conditional on being active, the lower margin determines $\pi^L_{i_n,j|A,t}$ across $j \in \mathcal{J}^L$.
- Hence the unconditional land share of active sector j is

$$\pi^L_{i_n,j,t} = \pi^L_{i_n,A,t}\pi^L_{i_n,j|A,t}.$$

- Physical land allocation is

$$F^{\text{Land}}_{i_n,j,t} = \pi^L_{i_n,j,t}\bar{F}_i^{\text{Land}}, \qquad F^{\text{Land}}_{i_n,0,t} = (1-\pi^L_{i_n,A,t})\bar{F}_i^{\text{Land}}.$$

- Land adds up exactly: $F^{\text{Land}}_{i_n,0,t} + \sum_{j\in\mathcal{J}^L} F^{\text{Land}}_{i_n,j,t} = \bar{F}_i^{\text{Land}}$.

Appendix: land payments and setup labor

Land Allocation: Two-Layer Outcomes

Define mean returns $R_{i_n,j,t} \equiv E_{i_n,j,t} c_{i_n,j,t} \hbar_{i_n,j,t}$ for $j \in \mathcal{J}^L$ and $R_{i_n,0,t} \equiv E_{i_n,0} w^{\mathrm{LAB}}_{n,t}$.

- **Lower nest — across active uses:**

$$R_{i_n,A,t} \equiv \left[\sum_{k \in \mathcal{J}^L} {R_{i_n,k,t}}^{\nu_2} \right]^{1/\nu_2}, \qquad \pi^L_{i_n,j|A,t} = \left(\frac{R_{i_n,j,t}}{R_{i_n,A,t}} \right)^{\nu_2}.$$

- **Upper nest — active versus idle:**

$$\pi^L_{i_n,A,t} = \frac{{R_{i_n,A,t}}^{\nu_1}}{{R_{i_n,0,t}}^{\nu_1} + {R_{i_n,A,t}}^{\nu_1}}, \qquad \pi^L_{i_n,0,t} = 1 - \pi^L_{i_n,A,t}.$$

- **Conditional efficiency:**

$$\mathbb{E}[e_{i_n,j,t}(\omega) \mid \omega \in \Omega_{i_n,j,t}] = E_{i_n,j,t} (\pi^L_{i_n,A,t})^{-1/\nu_1} (\pi^L_{i_n,j|A,t})^{-1/\nu_2}.$$

- **Grid output in a land-using sector:**

$$Q_{i_n,j,t} = \frac{\hbar_{i_n,j,t}}{\gamma^V_{n,j} s^L_{i_n,j,t}} \bar{F}^{\mathrm{Land}}_i E_{i_n,j,t} (\pi^L_{i_n,A,t})^{(\nu_1-1)/\nu_1} (\pi^L_{i_n,j|A,t})^{(\nu_2-1)/\nu_2}.$$

Appendix: return and output details

From Grid Prices to Armington Prices

$$c_{i_m,j,t} \longrightarrow P_{mm,j,t} \longrightarrow P_{nm,j,t} \longrightarrow P_{n,j,t}.$$

- Let $\bar{d}_{i_m,j} \geq 1$ denote the iceberg collection cost from grid i_m to country m's national hub. The national-hub price is total production value per unit of effective supply delivered to the hub:

$$P_{mm,j,t} \equiv \frac{\sum_{i\in\mathcal{I}_m} c_{i_m,j,t} Q_{i_m,j,t}}{Q_{m,j,t}}, \qquad Q_{m,j,t} \equiv \sum_{i\in\mathcal{I}_m} \frac{Q_{i_m,j,t}}{\bar{d}_{i_m,j}}.$$

- Equivalently, $P_{mm,j,t}$ is the **effective-quantity-weighted average** of $\bar{d}_{i_m,j} c_{i_m,j,t}$:

$$P_{mm,j,t} = \sum_{i\in\mathcal{I}_m} \frac{Q_{i_m,j,t}/\bar{d}_{i_m,j}}{Q_{m,j,t}} \bar{d}_{i_m,j} c_{i_m,j,t}, \qquad Y_{m,j,t} = P_{mm,j,t} Q_{m,j,t} = \sum_{i\in\mathcal{I}_m} c_{i_m,j,t} Q_{i_m,j,t}.$$

- Bilateral iceberg trade costs then map the national-hub price into the destination price:

$$P_{nm,j,t} = d_{nm,j} P_{mm,j,t}, \qquad d_{nn,j} = 1.$$

Appendix: collection value identity

Armington Trade: Two Nests

- **Foreign-origin nest.** Conditional on importing sector j, country n aggregates foreign origins $m \neq n$:

$$P^{\mathrm{Imp}}_{n,j,t} = \left[\sum_{m\neq n} \zeta^M_{nm,j} {P_{nm,j,t}}^{1-\sigma^M_j}\right]^{1/(1-\sigma^M_j)}, \qquad \lambda^M_{nm,j,t} = \frac{\zeta^M_{nm,j} {P_{nm,j,t}}^{1-\sigma^M_j}}{\sum_{\ell\neq n} \zeta^M_{n\ell,j} {P_{n\ell,j,t}}^{1-\sigma^M_j}}.$$

- **Domestic-versus-import nest.**

$$P_{n,j,t} = \left[\zeta^D_{n,j} {P_{nn,j,t}}^{1-\sigma^D_j} + (1-\zeta^D_{n,j})(P^{\mathrm{Imp}}_{n,j,t})^{1-\sigma^D_j}\right]^{1/(1-\sigma^D_j)}.$$

- Let $\lambda^D_{n,j,t}$ denote the domestic expenditure share. For $m \neq n$,

$$\lambda_{nm,j,t} = (1-\lambda^D_{n,j,t})\lambda^M_{nm,j,t}.$$

- σ^M_j governs substitution across foreign origins; σ^D_j governs substitution between the domestic and import composites.

Appendix: bilateral expenditure and absorption

Households

- Country n's representative household aggregates sectoral consumption using Cobb–Douglas preferences:

$$C_{n,t} = \prod_{j \in \mathcal{J}} C_{n,j,t}{}^{\alpha^C_{n,j}}, \qquad \sum_{j \in \mathcal{J}} \alpha^C_{n,j} = 1.$$

- The corresponding cost-of-living index is

$$P^C_{n,t} = \prod_{j \in \mathcal{J}} \left(\frac{P_{n,j,t}}{\alpha^C_{n,j}} \right)^{\alpha^C_{n,j}}, \qquad P_{n,j,t} C_{n,j,t} = \alpha^C_{n,j} P^C_{n,t} C_{n,t}.$$

- Households receive labor income, land revenue, water revenue, and an exogenous net transfer (trade deficit) $D_{n,t}$:

$$P^C_{n,t} C_{n,t} = w^{\mathrm{LAB}}_{n,t} \left(\bar{F}^{\mathrm{LAB}}_{n,t} - \sum_{i \in \mathcal{I}_n} S_{i_n,\mathrm{active},t} \right) + \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{J}^L} RL_{i_n,j,t} + \sum_{i \in \mathcal{I}_n} r^{\mathrm{Water}}_{i_n,t} \bar{F}^{\mathrm{Water}}_{i,t} + D_{n,t}.$$

where

$$S_{i_n,\mathrm{active},t} \equiv \sum_{j \in \mathcal{J}^L} \mathbb{E}[e_{i_n,0}(\omega) \mid \omega \in \Omega_{i_n,j,t}] \, F^{\mathrm{Land}}_{i_n,j,t}$$

is the setup/replacement requirement of active land, measured in labor units.

Market Clearing and Equilibrium

- **Goods:** origin-m sector-j output value equals worldwide expenditure on that origin:

$$Y_{m,j,t} = \sum_{n \in \mathcal{N}} X_{nm,j,t}, \qquad X_{nm,j,t} = \lambda_{nm,j,t} X_{n,j,t}.$$

- **Labor, water, and land:**

$$w_{n,t}^{\mathrm{LAB}} \bar{F}_{n,t}^{\mathrm{LAB}} = \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{J}} \gamma_{n,j}^{V} s_{i_n,j,t}^{\mathrm{LAB}} Y_{i_n,j,t} + w_{n,t}^{\mathrm{LAB}} \sum_{i \in \mathcal{I}_n} S_{i_n,\mathrm{active},t},$$

$$r_{i_n,t}^{\mathrm{Water}} \bar{F}_{i,t}^{\mathrm{Water}} = \sum_{j \in \mathcal{J}} \gamma_{n,j}^{V} s_{i_n,j,t}^{W} Y_{i_n,j,t}.$$

$$RL_{i_n,j,t} \equiv \left[\mathbb{E}\left[r_{i_n,j,t}^{\mathrm{Land}}(\omega) \mid \omega \in \Omega_{i_n,j,t} \right] \right] \pi_{i_n,j,t}^{L} \bar{F}_{i_n}^{\mathrm{Land}} = \gamma_{n,j}^{V} s_{i_n,j,t}^{L} Y_{i_n,j,t}.$$

- **Trade balance:**

$$\sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} X_{nm,j,t} = \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} X_{mn,j,t} + D_{n,t}, \qquad \sum_{n} D_{n,t} = 0.$$

Appendix: factor demands and accounting

Dynamic Extraction I: Revenue and Extraction Cost

- Water extraction generates local revenue

$$\mathrm{WaterRevenue}_{i,t} = r_{i_n,t}^{\mathrm{Water}} \bar{F}_{i,t}^{\mathrm{Water}} = r_{i_n,t}^{\mathrm{Water}} \mathcal{A}_i \mathsf{Div}_{i,t}.$$

- Extraction from source $a \in \{\mathrm{Sh}, \mathrm{Dp}\}$ has cost

$$\mathrm{ExtractCost}_i^a \left(\mathcal{A}_i \mathsf{Div}_{i,t}^a, \boldsymbol{W}_{ti}; G_i^a \right),$$

where $\boldsymbol{W}_{ti}$ is the local hydrologic state.

  - G_i^a captures source-specific extraction technology and efficiency improvements; better technology lowers extraction costs for a given volume and hydrologic state.

- Extraction costs therefore vary across sources and can rise as groundwater stocks become more depleted.

Dynamic Extraction II: Intertemporal Problem

- The resource owner chooses source-specific extraction paths to maximize the present value of water revenue net of extraction costs:

$$\max_{\{\mathrm{Div}^a_{i,t}\}_{i,t,a}} \sum_{t=0}^{T} \left(\prod_{\tau=1}^{t} \frac{1}{1+r^K_\tau} \right) \sum_{i \in \mathcal{I}} \left[r^{\mathrm{Water}}_{i_n,t} \mathcal{A}_i \mathrm{Div}_{i,t} - \sum_{a \in \{\mathrm{Sh},\mathrm{Dp}\}} \mathrm{ExtractCost}^a_i \left(\mathcal{A}_i \mathrm{Div}^a_{i,t}, \boldsymbol{W}_{ti}; G^a_i \right) \right].$$

- Extraction today changes future water stocks through the hydrologic law of motion.

- r^K_t is the opportunity cost used to discount future water rents.
  - The current model has no capital input, so r^K_t is taken as an exogenous discount rate.

- $\bar{F}^{\mathrm{Water}}_{i,t} \equiv \mathcal{A}_i \mathrm{Div}_{i,t}$ is endogenous across dates, although it is taken as given within each static equilibrium.

Appendix: state constraints

Dynamic Extraction III: Two No-Arbitrage Conditions

- For source $a \in \{\mathrm{Sh}, \mathrm{Dp}\}$, define marginal revenue, marginal extraction cost, and marginal net return:

$$MR^a_{i,t} \equiv \frac{\partial \left[r^{\mathrm{Water}}_{i_n,t} \mathcal{A}_i \sum_b \mathrm{Div}^b_{i,t} \right]}{\partial \mathrm{Div}^a_{i,t}}, \qquad NR^a_{i,t} \equiv MR^a_{i,t} - MC^a_{i,t},$$

  where $MC^a_{i,t}$ is the marginal extraction cost of source a.

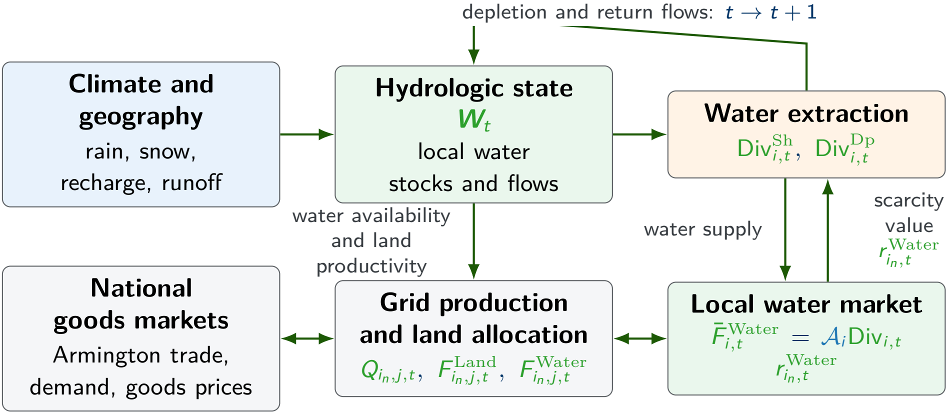
- **Across time — Hotelling no arbitrage:**

$$NR^a_{i,t+1} = \left(1 + r^K_{t+1}\right) NR^a_{i,t}, \qquad a \in \{\mathrm{Sh}, \mathrm{Dp}\}.$$

- **Across sources — within-period no arbitrage:**

$$NR^a_{i,t} = NR^b_{i,t}, \qquad a, b \in \{\mathrm{Sh}, \mathrm{Dp}\} \quad \text{if both are used.}$$

Hydrology and Economy: How the Model Works



Within a period, hydrologic states determine **water availability** and **land efficiency**, shaping **production costs and comparative advantage**; production, land allocation, trade, and demand determine **local water values**.

Across time, extraction and return flows change **water stocks**, shifting future **water availability and comparative advantage**.

Together, these margins govern **water use and broader economic activity over time**.

Appendix: Water States and Motion

- Let $s \in \{1, \ldots, 6\}$ index water states. For grid i,

$$\boldsymbol{W}_{ti} = \begin{pmatrix} W_{i,t}^{\mathrm{atm}} \\ W_{i,t}^{\mathrm{sur}} \\ W_{i,t}^{\mathrm{sub}} \\ W_{i,t}^{\mathrm{Sh}} \\ W_{i,t}^{\mathrm{Dp}} \\ W_{i,t}^{\mathrm{active}} \end{pmatrix} \in \mathbb{R}^6, \qquad \boldsymbol{W}_t \equiv [\boldsymbol{W}_{t1} \;\; \cdots \;\; \boldsymbol{W}_{tI}].$$

- Global water motion is

$$\boldsymbol{W}_t = \boldsymbol{W}_{t-1} + \Omega_t, \qquad \Omega_t = \mathbf{T}_t + \mathbf{S}_t,$$

where $\mathbf{T}_t$ contains within-grid transitions and $\mathbf{S}_t$ contains across-grid runoff routing.

← Back to Spatial Structure

Appendix: Within-Grid Transition Matrix

$$\mathbf{B}_{it} = \begin{pmatrix} 0 & r^{\text{rain}}_{i,t} + q_{i,t} + g_{i,t} & r^{\text{snow}}_{i,t} & 0 & 0 & r_{i,t} \\ e^{\text{open}}_{i,t} & 0 & 0 & 0 & 0 & \text{Div}^{\text{sur}}_{i,t} \\ e_{i,t} & q_{i,t} & 0 & f^{\text{Sh}}_{i,t} & 0 & \text{Div}^{\text{Sh}}_{i,t} \\ 0 & e_{i,t} & b_{i,t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{Div}^{\text{Dp}}_{i,t} \\ e_{i,t} & q_{i,t} & f_{i,t} & 0 & 0 & 0 \end{pmatrix}.$$

- $r_{i,t} = r^{\text{rain}}_{i,t} + r^{\text{snow}}_{i,t}$ is precipitation; $q_{i,t}$ is runoff; $f^{\text{Sh}}_{i,t}$ is shallow recharge; $b_{i,t}$ is baseflow.
- The withdrawal terms $\text{Div}^{\text{sur}}_{i,t}$, $\text{Div}^{\text{Sh}}_{i,t}$, and $\text{Div}^{\text{Dp}}_{i,t}$ enter the agricultural-land water column and link physical water motion to economic extraction.

← Back to Water Supply

Appendix: Exact CDE Primary-Factor Relation

The CDE primary-factor quantity index is defined implicitly by

$$1 = \beta_{n,j}^{\mathrm{LAB}} V_{i_n,j,t}{}^{\epsilon_j^{\mathrm{LAB}}(1-\sigma_j^{\mathrm{LAB}})} \left(\frac{w_{n,t}^{\mathrm{LAB}}}{P_{i_n,j,t}^{V} V_{i_n,j,t}} \right)^{1-\sigma_j^{\mathrm{LAB}}} + \beta_{n,j}^{L} V_{i_n,j,t}{}^{\epsilon_j^{L}(1-\sigma_j^{L})} \left(\frac{r_{i_n,j,t}^{\mathrm{Land}}(\omega)/e_{i_n,j,t}(\omega)}{P_{i_n,j,t}^{V} V_{i_n,j,t}} \right)^{1-\sigma_j^{L}} + \beta_{n,j}^{W} V_{i_n,j,t}{}^{\epsilon_j^{W}(1-\sigma_j^{W})} \left(\frac{r_{i_n,t}^{\mathrm{Water}}}{P_{i_n,j,t}^{V} V_{i_n,j,t}} \right)^{1-\sigma_j^{W}}.$$

- $\beta_{n,j}^{x}$ are distribution weights, σ_j^{x} are input-specific substitution parameters, and ϵ_j^{x} are expansion parameters for $x \in \{\mathrm{LAB}, L, W\}$.
- The plot index ω appears only through the land return/productivity ratio; labor and water prices are common within a grid-sector at date t.

← Back to Production

Appendix: CDE Shares and Grid-Sector Unit Cost

- Define primary-factor cost shares

$$s^{\mathrm{LAB}}_{i_n,j,t} = \frac{w^{\mathrm{LAB}}_{n,t} F^{\mathrm{LAB}}_{i_n,j,t}}{P^V_{i_n,j,t} V_{i_n,j,t}}, \quad s^L_{i_n,j,t} = \frac{RL_{i_n,j,t}}{P^V_{i_n,j,t} V_{i_n,j,t}}, \quad s^W_{i_n,j,t} = \frac{r^{\mathrm{Water}}_{i_n,t} F^{\mathrm{Water}}_{i_n,j,t}}{P^V_{i_n,j,t} V_{i_n,j,t}},$$

  with $s^{\mathrm{LAB}} + s^L + s^W = 1$.
- For the upper Cobb–Douglas nest, define

$$\Upsilon_{n,j} = (\gamma^V_{n,j})^{-\gamma^V_{n,j}} \prod_{k \in \mathcal{J}} {\gamma_{n,j,k}}^{-\gamma_{n,j,k}}.$$

  Then

$$c_{i_n,j,t} = \frac{\Upsilon_{n,j}}{A^{\mathrm{prod}}_{n,j,t}} (P^V_{i_n,j,t})^{\gamma^V_{n,j}} \prod_{k \in \mathcal{J}} (P_{n,k,t})^{\gamma_{n,j,k}}.$$

← Back to Production

Appendix: Land Return Implied by the CDE Relation

From collection zero profit,

$$c_{i_n,j,t} = \frac{P_{nn,j,t}}{\bar{d}_{i_n,j}}.$$

Solving the CDE relation for the return per efficiency unit of land defines $\hbar_{i_n,j,t}$:

$$\hbar_{i_n,j,t} \equiv \frac{P^V_{i_n,j,t} V_{i_n,j,t}}{c_{i_n,j,t}} \left[\frac{1 - \beta^{\mathrm{LAB}}_{n,j} V_{i_n,j,t}{}^{\epsilon^{\mathrm{LAB}}_j (1-\sigma^{\mathrm{LAB}}_j)} \left(\frac{w^{\mathrm{LAB}}_{n,t}}{P^V_{i_n,j,t} V_{i_n,j,t}} \right)^{1-\sigma^{\mathrm{LAB}}_j} - \beta^W_{n,j} V_{i_n,j,t}{}^{\epsilon^W_j (1-\sigma^W_j)} \left(\frac{r^{\mathrm{Water}}_{i_n,t}}{P^V_{i_n,j,t} V_{i_n,j,t}} \right)^{1-\sigma^W_j}}{\beta^L_{n,j} V_{i_n,j,t}{}^{\epsilon^L_j (1-\sigma^L_j)}} \right]^{1/(1-\sigma^L_j)}.$$

Hence

$$\frac{r^{\mathrm{Land}}_{i_n,j,t}(\omega)}{e_{i_n,j,t}(\omega)} = c_{i_n,j,t} \hbar_{i_n,j,t}, \qquad \mathbb{E}[r^{\mathrm{Land}}_{i_n,j,t}(\omega)] = E_{i_n,j,t} c_{i_n,j,t} \hbar_{i_n,j,t}.$$

← Back to Production

Appendix: Gross Land Payments and Setup Labor

- Let $\Omega_{i_n,j,t}$ be the set of plots selected into sector j. Gross active-land payment is

$$RL_{i_n,j,t} \equiv \mathbb{E}\left[r^{\mathrm{Land}}_{i_n,j,t}(\omega) \mid \omega \in \Omega_{i_n,j,t}\right] F^{\mathrm{Land}}_{i_n,j,t} = \gamma^{V}_{n,j} s^{L}_{i_n,j,t} Y_{i_n,j,t}.$$

- Activating land requires setup labor

$$S_{i_n,\mathrm{active},t} = E_{i_n,0} \bar{F}^{\mathrm{Land}}_{i} \left[1 - (\pi^{L}_{i_n,0,t})^{(\nu_1-1)/\nu_1}\right].$$

- Net land revenue is

$$NRL_{i_n,t} = \sum_{j\in\mathcal{J}^L} RL_{i_n,j,t} - w^{\mathrm{LAB}}_{n,t} S_{i_n,\mathrm{active},t}, \qquad NRL_{n,t} = \sum_{i\in\mathcal{I}_n} NRL_{i_n,t}.$$

← Back to Land Mapping

Appendix: Selection, Conditional Return, and Output

- Conditional selected productivity is

$$\mathbb{E}[e_{i_n,j,t}(\omega) \mid \omega \in \Omega_{i_n,j,t}] = E_{i_n,j,t}(\pi^L_{i_n,A,t})^{-1/\nu_1}(\pi^L_{i_n,j|A,t})^{-1/\nu_2}.$$

- Therefore conditional return is

$$\mathbb{E}[r^{\text{Land}}_{i_n,j,t}(\omega) \mid \omega \in \Omega_{i_n,j,t}] = R_{i_n,j,t}(\pi^L_{i_n,A,t})^{-1/\nu_1}(\pi^L_{i_n,j|A,t})^{-1/\nu_2}.$$

- Combining selected productivity, land size, and the CDE land cost share gives the output formula shown on the main slide.

← Back to Land Outcomes

Appendix: Collection-Hub Value Identity

For any active grid $i \in \mathcal{I}_m$,

$$P_{mm,j,t} = \bar{d}_{i_m,j} c_{i_m,j,t}.$$

Together with

$$Q_{m,j,t} = \sum_{i \in \mathcal{I}_m} \frac{Q_{i_m,j,t}}{\bar{d}_{i_m,j}},$$

this implies

$$Y_{m,j,t} = P_{mm,j,t} Q_{m,j,t} = \sum_{i \in \mathcal{I}_m} c_{i_m,j,t} Q_{i_m,j,t} = \sum_{i \in \mathcal{I}_m} Y_{i_m,j,t}.$$

Thus iceberg collection losses affect physical quantities delivered to the hub, while the value of national production equals the sum of grid production values.

← Back to Grid-to-Hub Trade

Appendix: Bilateral Expenditure and Absorption

- $Q_{nm,j,t}$ is destination-n absorption of origin-m sector-j goods and

$$X_{nm,j,t} = P_{nm,j,t} Q_{nm,j,t}.$$

- Total sector-j absorption and expenditure in destination n satisfy

$$X_{n,j,t} = \sum_{m \in \mathcal{N}} X_{nm,j,t} = P_{n,j,t} Q^{A}_{n,j,t}.$$

- Bilateral spending is

$$X_{nm,j,t} = \lambda_{nm,j,t} X_{n,j,t}.$$

← Back to Armington Trade

Appendix: Factor-Income Identity

- Production labor, gross land payments, water payments, and intermediates exhaust grid-sector output value:

$$Y_{i_n,j,t} = w^{\mathrm{LAB}}_{n,t} F^{\mathrm{LAB}}_{i_n,j,t} + RL_{i_n,j,t} + r^{\mathrm{Water}}_{i_n,t} F^{\mathrm{Water}}_{i_n,j,t} + \sum_{k \in \mathcal{J}} P_{n,k,t} M^{int}_{i_n,j,k,t}.$$

- Summing over sectors/grids, subtracting setup labor from gross land payments, and using goods-market clearing gives the household income identity on the main slide.
- This makes clear why setup labor is counted in national labor clearing but deducted when gross land payments are converted to net land income.

← Back to Households

Appendix: Factor Demands and Accounting

- The upper Cobb–Douglas nest assigns $\gamma^{V}_{n,j}$ of grid-sector output value to the primary-factor composite:

$$P^{V}_{i_n,j,t} V_{i_n,j,t} = \gamma^{V}_{n,j} Y_{i_n,j,t}.$$

- Direct labor, water, and intermediate demands satisfy

$$w^{\mathrm{LAB}}_{n,t} F^{\mathrm{LAB}}_{i_n,j,t} = \gamma^{V}_{n,j} s^{\mathrm{LAB}}_{i_n,j,t} Y_{i_n,j,t},$$

$$r^{\mathrm{Water}}_{i_n,t} F^{\mathrm{Water}}_{i_n,j,t} = \gamma^{V}_{n,j} s^{W}_{i_n,j,t} Y_{i_n,j,t}, \qquad P_{n,k,t} M^{int}_{i_n,j,k,t} = \gamma_{n,j,k} Y_{i_n,j,t}.$$

← Back to Equilibrium

Appendix: Extraction State Constraints

- Source-specific extraction is constrained by physical availability and enters the water-state transition system through the corresponding withdrawal terms in $\mathbf{B}_{it}$.
- In compact form, write the hydrologic state transition as

$$\boldsymbol{W}_{t+1} = \mathcal{G}\left(\boldsymbol{W}_t, \{\mathrm{Div}^a_{i,t}\}_{i,a}, \mathbf{Z}_t\right),$$

  where $\mathbf{Z}_t$ collects exogenous climate/geographic inputs.
- The dynamic problem additionally imposes

$$\mathrm{Div}^a_{i,t} \geq 0 \quad \text{and source-specific feasibility bounds implied by } \boldsymbol{W}_{ti}.$$

- The static economy is solved conditional on the current extracted-water endowment; the resulting $r^{\mathrm{Water}}_{i_n,t}$ feeds back into the extraction problem.

← Back to Intertemporal Extraction