Notes on DFS model

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1 Model without tariff

1.1 Environment

- Two countries: Home (H) and Foreign (F)
- A continuum of homogeneous goods $z \in [0, 1]$
- Labor is the only factor of production
- Country $i \in \{H, F\}$ is populated by L_i workers
- Each worker is paid a wage, w_i
- Perfect competition + constant returns to scale

1.2 Demand

The representative consumer in country $n \in \{H, F\}$ has a Cobb-Douglass utility

$$U_n(\mathbf{q}) = \int_0^1 b(z) \ln q(z)\, dz$$

- z indexes the good.
- $b(z)$ is the share of expenditure on good z
- By assumption: $\int_0^1 b(z)\, dz = 1$

Utility maximization implies

$$\begin{cases} p_H(z)q_H(z) = b(z)E_H \\ p_F(z)q_F(z) = b(z)E_F \end{cases}$$

- $p_i(z)q_i(z)$: expenditure on good z in country i.
- $\mathrm{Y}_i = w_iL_i$: total income in country i; $E_i = w_iL_i + D_i$: total expenditure

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1.3 Supply

- Let $a_i(z)$ denote the unit labor requirement for producing good z in country i.
- Order the goods so that $A(z) \equiv \frac{a_F(z)}{a_H(z)}$ is decreasing and *strictly monotone*.
  - H has a comparative advantage in the low-z goods.
  - F has a comparative advantage in the high-z goods.
- Suppose trade is costless: $p_H(z) = p_F(z) = p(z)$.
- Good z will be produced by H if

$$a_H(z)w_H < a_F(z)w_F \iff A(z) > \frac{w_H}{w_F}$$

- Good z will be produced by F if

$$a_H(z)w_H > a_F(z)w_F \iff A(z) < \frac{w_H}{w_F}$$

1.4 Equiliburium

Equilibrium Outcomes:

- relative wage $\omega = \frac{w_H}{w_F}$
- cut-off $\tilde{z}$, such that
  - H produces every good $z \in [0, \tilde{z}]$;
  - F produces every good $z \in [\tilde{z}, 1]$

Equilibrium Condition (1), from **production rule**:

$$A(\tilde{z}) = \omega$$

- Denote by $\theta(\tilde{z}) \equiv \int_0^{\tilde{z}} b(z)\, dz$ the fraction of income spent on goods produced in H.
- **Equilibrium Condition (2)**,from **balanced trade**:

$$\underbrace{\theta(\tilde{z})(w_F L_F + D_F)}_{\text{Home exports}} = \underbrace{[1 - \theta(\tilde{z})]\,(w_H L_H + D_H)}_{\text{Home imports}} + D_H$$

$$\underbrace{[1 - \theta(\tilde{z})]\,(w_H L_H + D_H)}_{\text{F exports}} = \underbrace{\theta(\tilde{z})(w_F L_F + D_F)}_{\text{F imports}} + D_F$$

[Balanced Trade]

$$\underbrace{\theta(\tilde{z})w_F L_F}_{\text{Home exports}} = \underbrace{[1 - \theta(\tilde{z})]\,w_H L_H}_{\text{Home imports}}$$

$$\omega = \frac{\theta(\tilde{z})}{1 - \theta(\tilde{z})}\left(\frac{L_F}{L_H}\right) \equiv B(\tilde{z})$$

- Note that $B(\cdot)$ is strictly increasing function, i.e., $B'(\cdot) > 0$
- Equilibrium conditions (1) and (2) jointly determine $(\tilde{z}, \omega)$

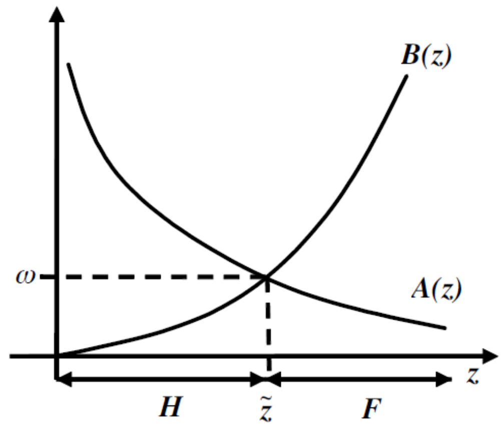

Figure 1:

1.5 Gains from trade

- Assign Home labor as the numeraire: $w_H = 1$
- After opening to trade
  - $Y_H = w_H L_H = L_H$ remains the same
  - $p_H(z)$ remains the same if z is not imported
  - $p_H(z)$ decreases if z is imported
- So, Home gains from trade!

1.6 Trade costs

- Until now, we assumed costless trade $\Longrightarrow p_H(z) = p_F(z)$
- Suppose trade is subject to an iceberg trade cost, τ:
  - Home will export good z if $\tau w_H a_H(z) \leq w_F a_F(z)$ (τ: 1 unit good from H to F, have to ship $\tau > 1$ units)
  - Foreign will export good z if $w_H a_H(z) \geq \tau w_F a_F(z)$ (τ: 1 unit good from F to H, have to ship $\tau > 1$ units)
- Define $\underline{z}$ such that: $\tau w_H a_H(\underline{z}) = w_F a_F(\underline{z})$
- Define $\bar{z}$ such that: $w_H a_H(\bar{z}) = \tau w_F a_F(\bar{z})$
  - Home will produce and export $z \in [0, \underline{z}]$
  - Foreign will produce and export $z \in [\bar{z}, 1]$
  - Goods $z \in [\underline{z}, \bar{z}]$ are non-traded.

- $$\underbrace{\theta(\underline{z})(w_F L_F + D_F)}_{\text{Home exports}} = \underbrace{\theta(\bar{z})(w_H L_H + D_H)}_{\text{Home imports}} + D_H$$

$$\underbrace{\theta(\bar{z})(w_H L_H + D_H)}_{\text{F exports}} = \underbrace{\theta(\underline{z})(w_F L_F + D_F)}_{\text{F imports}} + D_F$$

[Balanced Trade]

$$\omega = \frac{\theta(\underline{z})}{\theta(\bar{z})}\left(\frac{L_F}{L_H}\right) \equiv B(\tilde{z})$$

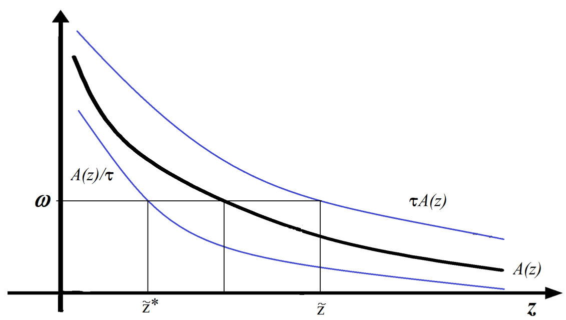

Figure 2:

2 Equilibrium

Table 1: Equilibrium Conditions under Free Trade

A1	$\omega \equiv \frac{w_H}{w_F}$; $A(z) \equiv \frac{a_F(z)}{a_H(z)}$
A2	$\omega = A(\tilde{z})$; $\theta(\tilde{z}) \equiv \int_0^{\tilde{z}} b(z)\,dz$
A3	$\omega = B(\tilde{z}) \equiv \frac{\theta(\tilde{z})}{1-\theta(\tilde{z})}\left(\frac{L_F}{L_H}\right)$;
Exp & Income	$E_i = w_i L_i + D_i$; $\mathrm{Y}_i = w_i L_i$; $D_i \equiv 0 \quad \forall i \in [H, F]$
Production Rule	$A(\tilde{z}) = \omega = \frac{w_H}{w_F}$
Trade Balance	$\underbrace{\theta(\tilde{z}) w_F L_F}_{\text{Home exports}} = \underbrace{[1-\theta(\tilde{z})]\, w_H L_H}_{\text{Home imports}}$

Table 2: Equilibrium Conditions under Iceberg Trade Cost

Home export trade cost	$\tau_{FH} > 1$	
Foreign export trade cost	$\tau_{HF} > 1$	
A1	$\omega \equiv \frac{w_H}{w_F}$; $A(z) \equiv \frac{a_F(z)}{a_H(z)}$	
A2	$\omega = \frac{A(\underline{z})}{\tau_{FH}}$; $\omega = \tau_{HF} A(\bar{z})$; $\theta(\underline{z}) \equiv \int_0^{\tilde{z}} b(z)\, dz$; $\theta(\bar{z}) \equiv \int_{\tilde{z}}^1 b(z)\, dz$	
A3	$\omega = B(\underline{z}, \bar{z}) \equiv \frac{\theta(\underline{z})}{\theta(\bar{z})} \left(\frac{L_F}{L_H} \right)$;	
Exp & Income	$E_i = w_i L_i + D_i$; $\mathrm{Y}_i = w_i L_i$; $D_i \equiv 0$	$\forall i \in [H, F]$
Production Rule	$\frac{A(\underline{z})}{\tau_{FH}} = \omega$; $\tau_{HF} A(\bar{z}) = \omega$	
Trade Balance	$\underbrace{\theta(\underline{z})(w_F L_F + D_F)}_{\text{Home exports}} = \underbrace{\theta(\bar{z})(w_H L_H + D_H)}_{\text{Home imports}} + D_H$	

3 Model with Trade cost

Table 3: Equilibrium Conditions under Iceberg Trade Cost and Tariff

Home export trade cost and ad valorem tariff	$\tau_{FH} > 1$; $1 + t_{fh}$; $\kappa_{FH} \equiv \tau_{FH}(1 + t_{fh})$
Foreign export trade cost and ad valorem tariff	$\tau_{HF} > 1$; $1 + t_{hf}$; $\kappa_{HF} \equiv \tau_{HF}(1 + t_{hf})$
(HF means home by from foreign)	
A1	$\omega \equiv \frac{w_H}{w_F}$; $A(z) \equiv \frac{a_F(z)}{a_H(z)}$
A2	$\omega = \frac{A(\underline{z})}{\kappa_{FH}}$; $\omega = \kappa_{HF} A(\bar{z})$; $\theta(\underline{z}) \equiv \int_0^{\tilde{z}} b(z)\, dz$; $\theta(\bar{z}) \equiv \int_{\tilde{z}}^1 b(z)\, dz$
A3	$\frac{\theta(\underline{z})}{\theta(\bar{z})} = \frac{w_H L_H + T_H}{w_F L_F + T_F}$;
A4	$T_H = \frac{t_{hf}}{1+t_{hf}} \int_{\tilde{z}}^1 b(z) w_F a_F(z) \kappa_{HF}\, dz$; $T_F = \frac{t_{fh}}{1+t_{fh}} \int_0^{\tilde{z}} b(z) w_H a_H(z) \kappa_{FH}\, dz$
Exp & Income	$E_i = w_i L_i + D_i + T_i$; $\mathrm{Y}_i = w_i L_i$; $D_i \equiv 0$
Tariff	$T_H = \frac{t_{hf}}{1+t_{hf}} \int_{\tilde{z}}^1 b(z) w_F a_F(z) \kappa_{HF}\, dz$; $T_F = \frac{t_{fh}}{1+t_{fh}} \int_0^{\tilde{z}} b(z) w_H a_H(z) \kappa_{FH}\, dz$
Production Rule	$\frac{A(\underline{z})}{\kappa_{FH}} = \omega$; $\kappa_{HF} A(\bar{z}) = \omega$
Trade Balance	$\underbrace{\theta(\underline{z})(w_F L_F + D_F + T_F)}_{\text{Home exports}} = \underbrace{\theta(\bar{z})(w_H L_H + D_H + T_H))}_{\text{Home imports}} + D_H$

- Given Home and Foreign technology described by $a(z)$ and $a^*(z)$, populations L, L^*, and preferences described by $b(z)$, an **equilibrium** consists of a threshold $\tilde{z}$ and wages w, w^* such that:

  1. Consumers maximize their utility: $b(z)$ is the share of expenditures on good z.
  2. Each country produces the range of goods for which it is the lowest-cost supplier:
     - Home produces goods in the range of $[0, \tilde{z}]$
     - Foreign produces goods in the range of $[\tilde{z}, 1]$
  3. Labor market clears in each country.

- The equilibrium is the intersection of A and B.
- The intersection is where the three conditions in our definition of equilibrium are all satisfied:
  - Conditions 1 and 3 are embedded in B.
  - Condition 2 is represented by A.

- The equilibrium objects are the threshold $\tilde{z}$ and the relative wage ω such that both of these values are simultaneously consistent with the consumption side (B) and with the production side (A).

Market clearing condition

- **Labor market clearing condition:**
  - The income (i.e. total wages) of Home equals what the world spends on goods produced at Home:
$$wL = v(\tilde{z})(wL + w^*L^*)$$
- Equivalent to the labor market clearing is the condition that trade is balanced.
  - Imports to Home equals Exports from Home:
$$(1 - v(\tilde{z}))wL = v(\tilde{z})w^*L^*$$
- If labor market clears in one country, it will also clear in the other country.

Tying the demand and supply

- Notice that:
  1. B represents the relation between relative wage and market clearing.
  2. Market clearing represents the relation between relative wage and expenditures.
- Hence, B represents the relation between relative wage and expenditures (consumer's expenditures).
- On the other hand, A represents the relation between relative wage and labor productivities (production technology).
- It is now the time to tie these parts together.