Demographic, Trade and Growth *

Author 1[†] Author 2[‡] Author 3[§]

1 Introduction

We study how demographic change shapes economic growth and international trade. We develop a dynamic multi-country model that combines overlapping-generations households, endogenous physical-capital accumulation, innovation, and Ricardian trade. Demographics affect the economy through three margins: the supply and age composition of effective labor, life-cycle saving and capital accumulation, and the allocation of workers to research and the resulting creation of new knowledge. Production uses labor, capital, and traded intermediate inputs, while country productivity evolves endogenously through age-dependent research activity. International trade links these forces across economies by changing market access, factor returns, operating profits, and the value of innovation.

We use the framework to study how differences in fertility, survival, participation, and population age structure generate persistent differences in growth and trade patterns, and how openness can amplify or mitigate the economic effects of demographic transition. Our quantitative application focuses on China and Japan, which provide contrasting stages of demographic transition, and asks how changes in population structure reshape saving, innovation, comparative advantage, trade, and long-run income. A central motivation is what we call the "China 2.0" question: as the demographic forces that supported China's earlier growth weaken, to what extent can capital accumulation, innovation, and international trade sustain productivity and income growth in the next stage of development?

2 Model

Time is discrete and extends indefinitely. The world economy consists of N countries, indexed by $i, n, \ell \in \{1, \dots, N\}$. Age groups are indexed by $g \in \mathcal{G} \equiv \{1, \dots, G\}$, and tradable varieties

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by $\omega \in [0, 1]$. At the beginning of period t, country i is characterized by its population distribution $\{N_{i,g,t}\}_{g \in \mathcal{G}}$, physical-capital stock $K_{i,t}$, and knowledge stock $\lambda_{i,t}$. Fertility, survival, and labor-force participation determine the exogenous demographic path, while physical capital and knowledge evolve endogenously.

Each firm has a production division and an R&D division. The production division uses labor, physical capital, and composite intermediate inputs to produce a continuum of varieties. At the beginning of each period, the best available productivity for each variety follows a Fréchet distribution determined by the country's knowledge stock. Varieties are traded internationally subject to iceberg trade costs. In each destination–variety market, the supplier with the lowest delivered cost serves the market and charges the delivered cost of the next-best supplier, following the Ricardian–Bertrand structure of Eaton and Kortum (2002) and Bernard et al. (2003).

The R&D division employs researchers from different age groups and generates new productivity points through a Poisson process. Research productivity varies with age, so the contribution of an age group to innovation depends on both its size and its research ability. New productivity points are added to the country's knowledge stock at the end of the period and become available for production in period $t + 1$. This timing separates current production from current research while allowing the value of future operating profits to determine firms' demand for researchers (Kortum, 1997; Eaton and Kortum, 1999, 2007; Santacreu, 2015; Cai et al., 2022). The production and R&D divisions are jointly owned by domestic households, which receive the firm's net payout.

Households live for at most G periods and face age-dependent survival probabilities. Conditional on labor-force participation, an individual of age g in country n supplies $l_{n,g}$ efficiency units of labor and can be assigned to either production or research. Households use labor income, capital income, and firm payouts to finance consumption and saving. Saving is invested in domestic physical capital and determines $K_{i,t+1}$, while research determines $\lambda_{i,t+1}$.

Demographic change affects the economy through three margins. It changes the supply of effective labor through participation and age-specific labor efficiency, the creation of knowledge through the age composition of researchers, and physical-capital accumulation through life-cycle saving. International trade connects these margins across countries by affecting prices, factor returns, market access, operating profits, and the value of new knowledge. The joint evolution of population, capital, and knowledge determines both the transition path and the stationary balanced-growth equilibrium (Ravikumar et al., 2019; Sposi et al., 2021; Sposi, 2022).

2.1 Demographics, participation, and labor allocation

Let $N_{n,g,t}$ denote the number of age-g households alive in country n at calendar date t. Households live for at most G periods. Let $f_{n,g,t}$ denote births per age-g household, and let $s_{n,g,t} \in [0,1]$ denote the probability of surviving from age $g-1$ to age g, with survival realized at calendar date t. In particular, $s_{n,1,t}$ is newborn survival and $s_{n,G+1,t} = 0$. Population evolves according to

$$N_{n,1,t+1} = s_{n,1,t} \sum_{g=1}^{G} f_{n,g,t} N_{n,g,t}, \tag{1}$$

$$N_{n,g+1,t+1} = s_{n,g+1,t+1} N_{n,g,t}, \qquad g = 1, \ldots, G-1. \tag{2}$$

Total population and the population share of age group g are

$$N_{n,t} \equiv \sum_{g=1}^{G} N_{n,g,t}, \qquad n_{n,g,t} \equiv \frac{N_{n,g,t}}{N_{n,t}}. \tag{3}$$

Thus, $n_{n,g,t}$ is the fraction of country n's population that is age g at date t, and

$$\sum_{g=1}^{G} n_{n,g,t} = 1. \tag{4}$$

The parameter $l_{n,g} \geq 0$ denotes the efficiency units supplied by one active age-g worker in country n. It captures both the country and age dimensions of effective labor and may combine education and human capital, the working-age profile, hours conditional on participation, and other age-specific productivity differences. The value $l_{n,g} = 0$ outside working ages is allowed. Let $\tau^L_{n,t} \in [0,1]$ be the fraction of potential workers outside the labor force, so $1 - \tau^L_{n,t}$ is the participation rate. Active age-g labor and its efficiency-unit counterpart are

$$L_{n,g,t} \equiv (1 - \tau^L_{n,t}) N_{n,g,t}, \qquad \mathcal{L}_{n,g,t} \equiv l_{n,g} L_{n,g,t}. \tag{5}$$

Conditional on participation, firms assign a fraction $s^R_{n,g,t} \in [0,1]$ of active age-g workers to research. The superscript R distinguishes the research-assignment share $s^R_{n,g,t}$ from the survival probability $s_{n,g,t}$. Researcher and production-worker headcounts are

$$L^R_{n,g,t} \equiv s^R_{n,g,t} L_{n,g,t}, \qquad L^P_{n,g,t} \equiv (1 - s^R_{n,g,t}) L_{n,g,t}, \tag{6}$$

and aggregate efficiency labor in the two activities is

$$\mathcal{L}^R_{n,t} \equiv \sum_{g=1}^{G} l_{n,g} L^R_{n,g,t}, \qquad \mathcal{L}^P_{n,t} \equiv \sum_{g=1}^{G} l_{n,g} L^P_{n,g,t}. \tag{7}$$

The competitive wage $W_{n,t}$ is paid per efficiency unit. An active age-g worker therefore earns $W_{n,t} l_{n,g}$, whether assigned to production or research. The representative age-g household receives average labor income

$$W_{n,t}(1-\tau^L_{n,t}) l_{n,g}. \tag{8}$$

Thus, the production–research assignment does not alter the household's labor income or its individual saving problem.

2.2 Production, trade, and operating profits

Country n combines a continuum of varieties into a nontradable composite good,

$$Q_{n,t} = \left[\int_0^1 x_{n,t}(\omega)^{\frac{\sigma-1}{\sigma}}\, \mathrm{d}\omega\right]^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1, \tag{9}$$

where $x_{n,t}(\omega)$ is total demand for variety ω. Let $P_{n,t}$ be the corresponding ideal price index and $X_{n,t} \equiv P_{n,t} Q_{n,t}$ nominal expenditure.

A producer in origin i with productivity $q_{i,t}(\omega)$ uses efficiency labor $h_{i,t}(\omega)$, physical capital $k_{i,t}(\omega)$, and the composite intermediate input $m_{i,t}(\omega)$:

$$y_{i,t}(\omega) = q_{i,t}(\omega) \left[h_{i,t}(\omega)^{\gamma_{L,i}} k_{i,t}(\omega)^{1-\gamma_{L,i}}\right]^{\gamma_i} m_{i,t}(\omega)^{1-\gamma_i}, \tag{10}$$

where $\gamma_i \gamma_{L,i}$, $\gamma_i(1-\gamma_{L,i})$, and $1-\gamma_i$ are the variable-cost shares of labor, capital, and intermediates. Let $R_{i,t}$ be the rental payment per unit of capital. Unit input cost before dividing by productivity is

$$c_{i,t} = \Upsilon_i \left(W_{i,t}^{\gamma_{L,i}} R_{i,t}^{1-\gamma_{L,i}}\right)^{\gamma_i} P_{i,t}^{1-\gamma_i}, \tag{11}$$

where Υ_i is the Cobb–Douglas cost constant defined in Appendix A.1.3. Shipping one unit from origin i to destination n requires $\kappa_{ni,t} \geq 1$ units at origin, so delivered unit cost is $\kappa_{ni,t} c_{i,t}/q$. Bilateral variables use destination–origin ordering throughout.

The best productivity available for a variety in country i is Fréchet:

$$\Pr[q_{i,t}(\omega) \leq q] = \exp[-\lambda_{i,t} q^{-\theta}], \qquad \theta > 0. \tag{12}$$

Define destination n's competitiveness index

$$\Phi_{n,t} \equiv \sum_{\ell=1}^{N} \lambda_{\ell,t}(c_{\ell,t}\kappa_{n\ell,t})^{-\theta}. \tag{13}$$

The lowest-cost supplier serves the market and charges the second-lowest delivered cost. The resulting bilateral expenditure share and price index are

$$\pi_{ni,t} \equiv \frac{X_{ni,t}}{X_{n,t}} = \frac{\lambda_{i,t}(c_{i,t}\kappa_{ni,t})^{-\theta}}{\Phi_{n,t}}, \tag{14}$$

$$P_{n,t} = \mathcal{B}(\theta,\sigma)\Phi_{n,t}^{-1/\theta}, \qquad \mathcal{B}(\theta,\sigma) \equiv \Gamma\left(2 - \frac{\sigma-1}{\theta}\right)^{\frac{1}{1-\sigma}}. \tag{15}$$

We impose $\theta > \sigma - 1$, which is sufficient for the price moment used in (15) to exist.

Country i's gross sales and operating profits are

$$Y_{i,t} \equiv \sum_{n=1}^{N} \pi_{ni,t}X_{n,t}, \qquad \Pi_{i,t}^{\mathrm{op}} = \frac{1}{1+\theta}Y_{i,t}. \tag{16}$$

The remaining fraction $\theta/(1+\theta)$ pays variable production costs. Factor payments therefore satisfy

$$W_{i,t}\mathcal{L}_{i,t}^{P} = \gamma_i\gamma_{L,i}\frac{\theta}{1+\theta}Y_{i,t}, \tag{17}$$

$$R_{i,t}K_{i,t} = \gamma_i(1-\gamma_{L,i})\frac{\theta}{1+\theta}Y_{i,t}, \tag{18}$$

$$P_{i,t}M_{i,t} = (1-\gamma_i)\frac{\theta}{1+\theta}Y_{i,t}, \tag{19}$$

where $M_{i,t}$ is aggregate intermediate demand. Only $\mathcal{L}_{i,t}^{P}$ enters current production; research labor affects output through future knowledge.

2.3 Research assignment and knowledge accumulation

Country i's current research productivity depends on the knowledge stock inherited at the beginning of period t:

$$\alpha_{i,t} = \bar{\alpha}_i\lambda_{i,t}^{\nu}, \qquad \bar{\alpha}_i > 0, \qquad 0 \le \nu < 1. \tag{20}$$

The parameter $\bar{\alpha}_i$ is country i's fundamental innovation efficiency. The term $\lambda_{i,t}^{\nu}$ captures the effect of existing knowledge on the productivity of current research.

Within the active workforce of age g, workers are ordered by $u \in (0,1]$. A rank-u worker contributes

$$b_{i,g,t}(u) = \alpha_{i,t}\eta_g \xi u^{\xi-1}, \qquad \eta_g > 0, \qquad 0 < \xi < 1, \tag{21}$$

to the Poisson arrival intensity of new ideas. The parameter η_g captures age-specific research ability. The parameter ξ governs the curvature of the within-age research schedule and hence diminishing returns to assigning a larger fraction of an age group to research. In contrast, $l_{i,g}$ measures the general labor efficiency of an age-g worker in country i and therefore determines the worker's wage per active person.

If a fraction $s^R_{i,g,t}$ of active age-g workers is assigned to research, the knowledge increment generated by this age group is

$$\begin{aligned}\Delta\lambda_{i,g,t} &= L_{i,g,t} \int_0^{s^R_{i,g,t}} b_{i,g,t}(u)\, \mathrm{d}u \\ &= \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g \left(s^R_{i,g,t}\right)^{\xi} L_{i,g,t}. \end{aligned} \tag{22}$$

Independent age-specific Poisson processes superimpose. Total new knowledge is therefore

$$\begin{aligned}\Delta\lambda_{i,t} &= \sum_{g=1}^{G} \Delta\lambda_{i,g,t} \\ &= \bar{\alpha}_i \lambda^{\nu}_{i,t} \sum_{g=1}^{G} \eta_g \left(s^R_{i,g,t}\right)^{\xi} L_{i,g,t}, \end{aligned} \tag{23}$$

and the knowledge stock evolves according to

$$\lambda_{i,t+1} = \lambda_{i,t} + \Delta\lambda_{i,t}. \tag{24}$$

Because every age group draws idea quality from the same Pareto kernel, age-specific Poisson superposition changes only the scale of the idea process. The accumulated productivity frontier therefore remains Fréchet. Appendix A.1.4 provides the stochastic foundation.

Let $v_{i,t}$ be the date-t nominal value of one unit of knowledge created during period t. A continuum of atomistic innovation firms chooses research assignments and takes $v_{i,t}$ and $W_{i,t}$ as given. Suppressing the firm index in a symmetric equilibrium, the age-g research problem is

$$\max_{0 \le s \le 1} L_{i,g,t} \left[v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g s^{\xi} - W_{i,t} l_{i,g} s \right]. \tag{25}$$

For an interior assignment, the first-order condition is

$$\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g \left(s^R_{i,g,t}\right)^{\xi-1} = W_{i,t} l_{i,g}. \tag{26}$$

The left-hand side is the value of the marginal knowledge increment generated by assigning one additional active age-g worker to research. The right-hand side is that worker's foregone production wage.

The research-assignment share is therefore

$$s^R_{i,g,t} = \min\left\{1, \left(\frac{\xi v_{i,t}\bar{\alpha}_i \lambda^{\nu}_{i,t}\eta_g}{W_{i,t}l_{i,g}}\right)^{\frac{1}{1-\xi}}\right\}, \qquad l_{i,g} > 0. \tag{27}$$

The participation factor $1-\tau^L_{i,t}$ multiplies the number of active workers $L_{i,g,t}$ in both the value and wage-bill terms of (25); it therefore cancels from the private first-order condition. Participation has no direct effect on $s^R_{i,g,t}$, but it changes aggregate research employment, current production labor, knowledge accumulation, wages, and $v_{i,t}$, and can therefore affect the research assignment indirectly in general equilibrium.

2.4 OLG households and endogenous physical capital

In each country n, households live in an overlapping-generations environment with exogenous fertility and survival. A household born at calendar date t is age g at calendar date $t+g-1$. Its consumption and asset choices are therefore indexed as

$$\left\{c_{n,g,t+g-1}\right\}_{g=1}^{G}, \qquad \left\{a_{n,g+1,t+g}\right\}_{g=1}^{G-1}.$$

Here $a_{n,g,t+g-1} \geq 0$ is the physical-capital claim held at the beginning of age g, and $c_{n,g,t+g-1} > 0$ is consumption at that age.

Period utility is CRRA:

$$u(c) = \begin{cases} \dfrac{c^{1-1/\rho}}{1-1/\rho}, & \rho \neq 1, \\ \log c, & \rho = 1, \end{cases} \qquad \rho > 0. \tag{28}$$

Let $\beta \in (0,1)$ be the household discount factor. The term $\psi_{n,\tau} > 0$ is a saving wedge associated with assets that pay off in calendar date τ. A larger $\psi_{n,\tau}$ raises the incentive to carry resources into date τ and is calibrated to match the observed capital stock. The initial value $\psi_{n,1}$ is calibrated from the initial steady state, while the transition path $\{\psi_{n,t}\}_{t=2}^{T+1}$ is calibrated to the observed capital path. For a household born at date t, define the unconditional probability of surviving to age g by

$$S_{n,g,t+g-1} \equiv \prod_{h=2}^{g} s_{n,h,t+h-1}, \qquad S_{n,1,t} = 1. \tag{29}$$

The household chooses lifetime consumption and saving to maximize

$$\max_{\{c_{n,g,t+g-1}\}_{g=1}^{G},\{a_{n,g+1,t+g}\}_{g=1}^{G-1}} \quad \sum_{g=1}^{G} \beta^{g-1} S_{n,g,t+g-1} u\left(c_{n,g,t+g-1}\right). \tag{30}$$

Let $D^F_{n,\tau}$ denote the aggregate net payout of the consolidated domestic corporate sector and

$$d^F_{n,\tau} \equiv \frac{D^F_{n,\tau}}{N_{n,\tau}}$$

its per-capita distribution. Let $tr^D_{n,\tau}$ denote the baseline equal-per-capita demographic transfer, including redistributed accidental bequests, and let $tr^T_{n,\tau}$ denote a per-capita international transfer. At age g, calendar date $\tau = t + g - 1$, the budget constraint is

$$\begin{aligned} P_{n,t+g-1}c_{n,g,t+g-1} + P_{n,t+g-1}a_{n,g+1,t+g} = {} & P_{n,t+g-1}\left(1 + r_{n,t+g-1}\right) a_{n,g,t+g-1} \\ & + W_{n,t+g-1}\left(1 - \tau^L_{n,t+g-1}\right) l_{n,g} \\ & + d^F_{n,t+g-1} + tr^D_{n,t+g-1} + tr^T_{n,t+g-1}, \end{aligned} \tag{31}$$

for $g = 1, \dots, G$, with

$$a_{n,1,t} = 0, \qquad a_{n,G+1,t+G} = 0, \qquad c_{n,g,t+g-1} > 0. \tag{32}$$

Let $r_{n,\tau}$ denote the net real return on physical capital paid at calendar date τ:

$$r_{n,\tau} \equiv \frac{R_{n,\tau}}{P_{n,\tau}} - \delta, \tag{33}$$

where $\delta \in (0, 1)$ is the physical-capital depreciation rate. Hence the gross real payoff on one unit of capital carried into date τ is $1 + r_{n,\tau}$.

Equivalently, for an age-g household observed at calendar date τ, the recursive problem is

$$V_{n,g,\tau}(a) = \max_{c,a' \geq 0} \left\{ u(c) + \beta \psi_{n,\tau+1} s_{n,g+1,\tau+1} V_{n,g+1,\tau+1}(a') \right\}, \tag{34}$$

subject to the calendar-time version of (31), with $V_{n,G+1,\tau} = 0$.

For an unconstrained saver, the Euler equation for a household born at date t is

$$u'\left(c_{n,g,t+g-1}\right) = \beta \psi_{n,t+g} s_{n,g+1,t+g} \left(1 + r_{n,t+g}\right) u'\left(c_{n,g+1,t+g}\right), \qquad g = 1, \dots, G-1. \tag{35}$$

At any calendar date t, aggregate consumption and physical capital are

$$C_{n,t} \equiv \sum_{g=1}^{G} N_{n,g,t} c_{n,g,t}, \tag{36}$$

$$K_{n,t} \equiv \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}, \tag{37}$$

$$K_{n,t+1} \equiv \sum_{g=1}^{G-1} N_{n,g,t} a_{n,g+1,t+1}. \tag{38}$$

Gross investment is

$$I_{n,t} \equiv K_{n,t+1} - (1-\delta) K_{n,t}. \tag{39}$$

Capital is internationally immobile in the baseline, so domestic life-cycle saving determines domestic physical capital.

2.5 Corporate payouts and the value of knowledge

Variety production generates $\Pi^{\text{op}}_{i,t}$, while innovation firms pay the research wage bill $W_{i,t}\mathcal{L}^R_{i,t}$. The two activities may be legally separate firms connected through licensing, or the production and R&D divisions of an integrated firm. Any royalty or licensing payment between them cancels upon consolidation. The net corporate payout to domestic households is therefore

$$D^F_{i,t} = \Pi^{\text{op}}_{i,t} - W_{i,t}\mathcal{L}^R_{i,t}. \tag{40}$$

If $D^F_{i,t} < 0$, households make a net equity contribution to the corporate sector rather than receiving a positive dividend.

A unit of knowledge created during t first earns operating profits in $t+1$. Poisson marking implies that one unit of country i's knowledge stock receives the expected flow $\Pi^{\text{op}}_{i,t+1}/\lambda_{i,t+1}$. Existing claims to knowledge profits and physical capital are tradable domestically. No-arbitrage therefore discounts the knowledge payoff at the endogenous capital return:

$$v_{i,t} = \frac{P_{i,t}}{P_{i,t+1}(1+r_{i,t+1})} \left[\frac{\Pi^{\text{op}}_{i,t+1}}{\lambda_{i,t+1}} + v_{i,t+1} \right]. \tag{41}$$

Appendix A.1.10 derives the Poisson-marking allocation, the present-value representation, and the no-bubble condition.

2.6 Aggregation and equilibrium dynamics

2.6.1 Aggregation

Let $D_{i,t}$ be country i's trade deficit, with $\sum_i D_{i,t} = 0$, and set $tr^T_{i,t} = D_{i,t}/N_{i,t}$. Aggregating household budgets and redistributing accidental bequests yields the *unconsolidated* household resource identity

$$\begin{aligned} P_{i,t}C_{i,t} + P_{i,t}I_{i,t} = W_{i,t}\mathcal{L}^P_{i,t} + W_{i,t}\mathcal{L}^R_{i,t} + R_{i,t}K_{i,t} \\ + D^F_{i,t} + D_{i,t}. \end{aligned} \tag{42}$$

Equation (42) displays researcher income explicitly: $W_{i,t}\mathcal{L}^R_{i,t}$ is received by households as labor income. Substituting the corporate payout (40) gives the consolidated identity

$$P_{i,t}C_{i,t} + P_{i,t}I_{i,t} = W_{i,t}\mathcal{L}^P_{i,t} + R_{i,t}K_{i,t} + \Pi^{\mathrm{op}}_{i,t} + D_{i,t}. \tag{43}$$

Research wages have not been omitted: they appear once as household income and once as a corporate expense and cancel only after consolidation.

The composite-good resource constraint is

$$Q_{i,t} = C_{i,t} + I_{i,t} + M_{i,t}, \qquad X_{i,t} = P_{i,t}Q_{i,t} = Y_{i,t} + D_{i,t}. \tag{44}$$

Original aggregation equations retained from Section 3.4.1. Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

Total demand of capital and labor:

$$\sum_{j=1}^{J} \int_0^1 l^j_{n,t}(\omega)d\omega = \mathcal{L}^P_{n,t} \tag{45}$$

$$\sum_{j=1}^{J} \int_0^1 k^j_{n,t}(\omega)d\omega = K_{n,t} \tag{46}$$

Now, aggregate individual variables across cohorts, I have:

$$\begin{aligned} C_{n,t} &\equiv \sum_{g=1}^{G} N_{n,g,t}c_{n,g,t} \\ K_{n,t} &\equiv \sum_{g=2}^{G} N_{n,g-1,t-1}a_{n,g,t} \end{aligned} \tag{47}$$

Aggregating individual budget constraints across ages. the budget constraint at the aggregate level is

$$P_{n,t}\sum_{g=1}^{G}N_{n,g,t}c_{n,g,t}+P_{n,t}\sum_{g=1}^{G}N_{n,g,t}a_{n,g+1,t+1}=P_{n,t}\left(1+r_{n,t}\right)\sum_{g=1}^{G}N_{n,g,t}a_{n,g,t}$$
$$+\sum_{g=1}^{G}N_{n,g,t}W_{n,t}\left(1-\tau_{n,t}^{L}\right)l_{n,g}+\sum_{g=1}^{G}N_{n,g,t}tr_{n,t}^{D}+\sum_{g=1}^{G}N_{n,g,t}tr_{n,t}^{T} \tag{48}$$

Aggregate investment is defined as

$$I_t \equiv K_{t+1}-(1-\delta)\,K_t. \tag{49}$$

Equation 48 and Equation 49 implies:

$$P_{n,t}C_{n,t}+P_{n,t}I_{n,t}=R_{n,t}K_{n,t}+L_{n,t}^{e}W_{n,t}+D_{n,t},\ \text{with}\ R_{n,t}=(r_{n,t}+\delta)\,P_{n,t}. \tag{50}$$

Finally, each composite good is used both as a final good (consumption and investment) and as an intermediate input. Total expenditure on the composite good in economy n at time t is

$$X_{n,t}=P_{n,t}\,C_{n,t}+P_{n,t}\,I_{n,t}+(1-\gamma_n)Y_{n,t}, \tag{51}$$

where $P_{n,t}\,C_{n,t}+P_{n,t}\,I_{n,t}$ is final demand in economy n, and the last term is intermediate demand. In particular, $(1-\gamma_n)$ denotes the expenditure share of the composite good in the input bundle used by producers in economy n, and total spending on intermediates equals total revenue $Y_{n,t}=\sum_{i=1}^{N}X_{in,t}$.

2.6.2 Equilibrium dynamics

Definition 1 (Transitional equilibrium) *Given initial knowledge and age-specific assets, $\{\lambda_{n,0},a_{n,g,0}\}_{n,g}$, together with demographic paths, participation paths, trade costs, trade-deficit paths, preference shifters, and model parameters, a transitional equilibrium is a sequence of demographic allocations, household choices, research assignments, capital and knowledge stocks, prices, trade flows, and corporate payouts satisfying: household optimality; innovation-firm optimality; the demographic, capital, and knowledge laws of motion; production and trade; labor, capital, and goods-market clearing; corporate and aggregate accounting; one nominal normalization each period; and the knowledge-value no-bubble condition.*

2.7 Balanced growth and the stationary steady state

Suppose that, after date T, all countries share a constant population growth rate g_n, stable age shares, constant survival and fertility rates, constant participation, stationary research-assignment shares, constant trade costs and wedges, and a common gross-output labor share

$$\alpha_L \equiv \gamma_i \gamma_{L,i}. \tag{52}$$

A balanced growth path is a transitional equilibrium in which trade shares, factor shares, the net real capital return, and age profiles after removing their common growth trends are constant.

Proposition 1 (Balanced-growth rates) *On a balanced growth path with interior stationary research-assignment shares, the growth rates of knowledge, the productivity frontier, and real per-capita quantities satisfy*

$$1 + g_\lambda = (1 + g_n)^{\frac{1}{1-\nu}}, \tag{53}$$

$$1 + g_A = (1 + g_\lambda)^{1/\theta}, \tag{54}$$

$$1 + g_\omega = (1 + g_A)^{1/\alpha_L} = (1 + g_n)^{\frac{1}{\theta\alpha_L(1-\nu)}}. \tag{55}$$

Individual consumption, individual assets, and the real wage grow at $1 + g_\omega$*. Aggregate consumption, capital, investment, output, and real expenditure grow at*

$$1 + g_K = (1 + g_n)(1 + g_\omega). \tag{56}$$

The net real capital return and bilateral trade shares are stationary. If $g_n = 0$*, then*

$$g_\lambda = g_A = g_\omega = g_K = 0,$$

and the balanced growth path becomes a stationary steady state.

The balanced-growth capital level is endogenous. It is jointly pinned down by the age-specific household Euler equations, age-specific budget constraints, asset boundary conditions, accidental-bequest transfers, and capital aggregation. Appendix A.1.13 proves the proposition and reports the complete stationary system.

Table 1: Equilibrium system: transition path and SS/BGP

Block	Transition path (TS)	Steady state or balanced growth (SS/BGP)
Demographics	$N_{n,g,t}$ follows (1)–(2); age shares and participation may vary.	Stable age shares; $N_{n,g,t+1} = (1+g_n)N_{n,g,t}$; $\tau^L_{n,t} = \tau^L_{n,T}$.
Labor allocation	$L_{n,g,t} = (1-\tau^L_{n,t})N_{n,g,t}$; $L^R_{n,g,t} = s^R_{n,g,t}L_{n,g,t}$; $L^P_{n,g,t} = (1-s^R_{n,g,t})L_{n,g,t}$.	Age-specific $s^R_{n,g,t}$ and the production and research assignments are constant.
Research cutoff	$\xi v_{n,t}\bar{\alpha}_n\lambda^{\nu}_{n,t}\eta_g(s^R_{n,g,t})^{\xi-1} = W_{n,t}l_{n,g}$. The participation factor does not enter directly.	$(v_{n,t}/P_{n,t})\lambda^{\nu}_{n,t}/(W_{n,t}/P_{n,t})$ is constant, implying constant interior $s^R_{n,g,t}$.
Knowledge	$\lambda_{n,t+1} = \lambda_{n,t} + \bar{\alpha}_n\lambda^{\nu}_{n,t}\sum_g \eta_g(s^R_{n,g,t})^{\xi}L_{n,g,t}$.	$\lambda_{n,t+1}/\lambda_{n,t} = 1+g_\lambda = (1+g_n)^{1/(1-\nu)}$.
Households	Age-specific budgets, Euler equations, survival, transfers, and borrowing constraints determine $c_{n,g,t}$ and $a_{n,g+1,t+1}$.	$c_{n,g,t+1} = (1+g_\omega)c_{n,g,t}$ and $a_{n,g,t+1} = (1+g_\omega)a_{n,g,t}$; the age profiles satisfy the stationary Euler and budget systems.
Capital	$K_{n,t+1} = \sum_{g<G} N_{n,g,t}a_{n,g+1,t+1}$; $I_{n,t} = K_{n,t+1} - (1-\delta)K_{n,t}$.	$K_{n,t+1}/K_{n,t} = 1+g_K = (1+g_n)(1+g_\omega)$; the capital level clears the domestic asset market.
Trade and prices	Equations (11)–(19) hold at each date.	Trade shares, factor shares, and $R_{n,t}/P_{n,t}$ are constant; real per-capita quantities grow at $1+g_\omega$.
Corporate sector	$D^F_{n,t} = \Pi^{\text{op}}_{n,t} - W_{n,t}\mathcal{L}^R_{n,t}$; $v_{n,t}$ satisfies (41).	The value of knowledge satisfies the stationary recursion and the no-bubble condition.
Accounting	Both (42) and (43) hold.	Their balanced-growth counterparts hold with aggregate growth $1+g_K$.

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A open economy model details

This appendix provides the complete demographic, household, production, trade, innovation, valuation, accounting, balanced-growth, and numerical equilibrium systems underlying Section 2. We use $L_{i,g,t}$ for active worker headcount and $\mathcal{L}_{i,g,t} = l_{i,g} L_{i,g,t}$ for efficiency labor. The parameter $l_{i,g}$ captures both country-specific and age-specific effective labor, including education and human capital, the working-age profile, hours conditional on participation, and other age-specific productivity differences. This distinction is important: the gross cost of assigning one active age-g worker is $W_{i,t} l_{i,g}$, while the participation rate determines how many workers are available.

A.1 Model details and/or equation derivations

A.1.1 Age structure

The implied unconditional probability of surviving to age g at calendar date t is

$$S_{n,g,t} = \prod_{k=1}^{g} s_{n,k,t+k-g}. \tag{57}$$

The unconditional probability of dying at age g at date t is

$$D_{n,g,t} = S_{n,g-1,t-1}\left(1 - s_{n,g,t}\right) = S_{n,g-1,t-1} - S_{n,g,t}. \tag{58}$$

The expected life expectancy of individuals born at date t is

$$\bar{g}_{n,t} = \sum_{k=1}^{G} kD_{n,k,t+k-1}. \tag{59}$$

At date t, if the demographic process has reached a steady state or a balanced growth path, the population share of age group g is constant and is denoted by $n_{n,g}$. Therefore,

$$N_{n,g,t} = n_{n,g}N_{n,t}, \qquad \sum_{g=1}^{G} n_{n,g} = 1. \tag{60}$$

The demographic process is

$$N_{n,1,t+1} = s_{n,1,t+1}\sum_{g=1}^{G} f_{n,g,t}N_{n,g,t}, \tag{61}$$

$$N_{n,g+1,t+1} = s_{n,g+1,t+1}N_{n,g,t}. \tag{62}$$

Equivalently,

$$\begin{bmatrix} N_{n,1,t+1} \\ \vdots \\ N_{n,g,t+1} \\ \vdots \\ N_{n,G,t+1} \end{bmatrix} = \begin{bmatrix} s_{n,1,t+1}f_{n,1,t} & \cdots & s_{n,1,t+1}f_{n,g,t} & \cdots & s_{n,1,t+1}f_{n,G,t} \\ s_{n,2,t+1} & 0 & 0 & \cdots & 0 \\ 0 & s_{n,3,t+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n,G,t+1} & 0 \end{bmatrix} \begin{bmatrix} N_{n,1,t} \\ \vdots \\ N_{n,g,t} \\ \vdots \\ N_{n,G,t} \end{bmatrix}. \tag{63}$$

Writing $\mathbf{N}_{n,t} = (N_{n,1,t}, \ldots, N_{n,G,t})'$, this system is

$$\mathbf{N}_{n,t+1} = \mathbf{\Omega}_{n,t}\mathbf{N}_{n,t}. \tag{64}$$

Demography and labor-force participation Let

$$\mathbf{N}_{n,t} \equiv (N_{n,1,t}, \ldots, N_{n,G,t})'.$$

The demographic laws (1)–(2) can be written as

$$\mathbf{N}_{n,t+1} = \mathbf{\Omega}_{n,t}\mathbf{N}_{n,t}, \tag{65}$$

where the first row of $\mathbf{\Omega}_{n,t}$ is $\{s_{n,1,t}f_{n,g,t}\}_{g=1}^{G}$, and the element in row $g+1$, column g is $s_{n,g+1,t+1}$ for $g = 1, \ldots, G-1$. If demographic rates become constant, define the stable population-share vector

$$\mathbf{n}_n \equiv (n_{n,1}, \ldots, n_{n,G})'. \tag{66}$$

Let g_n denote the population growth rate. The stable population shares satisfy

$$\mathbf{\Omega}_n\mathbf{n}_n = (1+g_n)\mathbf{n}_n, \qquad \mathbf{1}'\mathbf{n}_n = 1. \tag{67}$$

For a household born at date t, the unconditional probability of surviving to age g, reached at calendar date $t+g-1$, is

$$S_{n,g,t+g-1} \equiv \prod_{h=2}^{g} s_{n,h,t+h-1}, \qquad S_{n,1,t} = 1. \tag{68}$$

This definition is consistent with $s_{n,g,\tau}$ being the probability of surviving from age $g-1$ to age g at calendar date τ.

The country–age efficiency profile $l_{n,g}$ is primitive. It measures the effective labor units supplied by an active age-g worker in country n and may combine education and human capital, the working-age profile, hours conditional on participation, and other age-specific productivity differences. The participation margin is kept separate:

$$L_{n,g,t} = (1-\tau^{L}_{n,t})N_{n,g,t}, \qquad \mathcal{L}_{n,g,t} = l_{n,g}(1-\tau^{L}_{n,t})N_{n,g,t}. \tag{69}$$

The decomposition allows demographic composition, participation, and efficiency to be changed independently in counterfactuals.

Conditional on participation,

$$L^{R}_{n,g,t} = s^{R}_{n,g,t}L_{n,g,t}, \qquad L^{P}_{n,g,t} = (1-s^{R}_{n,g,t})L_{n,g,t}, \tag{70}$$

$$\mathcal{L}^{R}_{n,t} = \sum_{g=1}^{G} l_{n,g}L^{R}_{n,g,t}, \qquad \mathcal{L}^{P}_{n,t} = \sum_{g=1}^{G} l_{n,g}L^{P}_{n,g,t}. \tag{71}$$

Hence

$$\mathcal{L}_{n,t}^{R} + \mathcal{L}_{n,t}^{P} = (1 - \tau_{n,t}^{L}) \sum_{g=1}^{G} l_{n,g} N_{n,g,t}. \tag{72}$$

The within-period timing is as follows. Countries enter period t with $\lambda_{n,t}$, $K_{n,t}$, and age-specific assets $\{a_{n,g,t}\}_{g=1}^{G}$. Active workers are realized and firms choose research assignments. Production, trade, factor prices, and operating profits are determined jointly with these assignments. Households consume and choose $\{a_{n,g+1,t+1}\}_{g=1}^{G-1}$. Research generates new productivity points. Household saving becomes $K_{n,t+1}$, and new ideas become productive through $\lambda_{n,t+1}$.

A.1.2 Financial market

The financial market receives deposits of $a_{n,g,t}P_{n,t}$ from individuals and must repay those individuals an amount $a_{n,g,t}(1 + r_{n,t})P_{n,t}$. The financial market lends

$$K_{n,t} = \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}$$

to firms for use in production. Firms return

$$\left(1 + \frac{R_{n,t}}{P_{n,t}} - \delta\right) P_{n,t} K_{n,t}$$

to the financial market. This expression incorporates physical-capital depreciation, so the amount returned is reduced by the depreciated capital. To clear the financial market, deposits and repayments must satisfy

$$\sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t} (1 + r_{n,t}) P_{n,t} = \left(1 + \frac{R_{n,t}}{P_{n,t}} - \delta\right) P_{n,t} K_{n,t}.$$

Because the baseline is financially autarkic,

$$K_{n,t} = \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}.$$

Therefore,

$$r_{n,t} = \frac{R_{n,t}}{P_{n,t}} - \delta.$$

Preferences, survival, and the individual budget The saving wedge $\psi_{n,\tau} > 0$ applies to assets that pay off at calendar date τ. It enters the Euler equation directly and is calibrated to match the steady-state and transition paths of aggregate capital.

Consider a household born in country n at calendar date t. It is age g at calendar date $t+g-1$. Its lifetime problem is

$$\max_{\{c_{n,g,t+g-1}\}_{g=1}^{G},\{a_{n,g+1,t+g}\}_{g=1}^{G-1}} \quad \sum_{g=1}^{G} \beta^{g-1} S_{n,g,t+g-1} u\left(c_{n,g,t+g-1}\right), \tag{73}$$

where $S_{n,g,t+g-1}$ is defined in (68).

At age g, the household satisfies

$$\begin{aligned} P_{n,t+g-1}c_{n,g,t+g-1} + P_{n,t+g-1}a_{n,g+1,t+g} = {} & P_{n,t+g-1}\left(1+r_{n,t+g-1}\right)a_{n,g,t+g-1} \\ & + W_{n,t+g-1}(1-\tau^{L}_{n,t+g-1})l_{n,g} \\ & + d^{F}_{n,t+g-1} + tr^{D}_{n,t+g-1} + tr^{T}_{n,t+g-1}. \end{aligned} \tag{74}$$

The baseline transfers accidental bequests equally across households, as in the original OLG specification. A more general age-specific redistribution rule can be represented by replacing $tr^{D}_{n,t}$ with $tr^{D}_{n,g,t}$, subject to the same aggregate transfer-clearing condition.

Newborn and terminal assets satisfy

$$a_{n,1,t} = 0, \qquad a_{n,G+1,t+G} = 0. \tag{75}$$

The household receives average age-cell labor income. Since every active worker of age g is paid $W_{n,\tau}l_{n,g}$, its budget is identical whether the active member is assigned to production or research.

The equivalent recursive problem for an age-g household observed at calendar date τ is

$$\begin{aligned} V_{n,g,\tau}(a_{n,g,\tau}) = \max_{c_{n,g,\tau},\, a_{n,g+1,\tau+1}\geq 0} \Big\{ & u(c_{n,g,\tau}) \\ & + \beta\psi_{n,\tau+1}s_{n,g+1,\tau+1}V_{n,g+1,\tau+1}\left(a_{n,g+1,\tau+1}\right)\Big\}, \end{aligned} \tag{76}$$

with $V_{n,G+1,\tau} = 0$.

Euler equation and borrowing constraint Let $\mu_{n,g,\tau}$ be the multiplier on the budget constraint and $\chi_{n,g,\tau} \geq 0$ the multiplier on $a_{n,g+1,\tau+1} \geq 0$. The first-order conditions are

$$u'(c_{n,g,\tau}) = \mu_{n,g,\tau} P_{n,\tau}, \tag{77}$$

$$\mu_{n,g,\tau} P_{n,\tau} = \beta \psi_{n,\tau+1} s_{n,g+1,\tau+1} \mu_{n,g+1,\tau+1} P_{n,\tau+1} \left(1 + r_{n,\tau+1}\right) + \chi_{n,g,\tau}, \tag{78}$$

$$\chi_{n,g,\tau} a_{n,g+1,\tau+1} = 0. \tag{79}$$

The net real capital return is

$$r_{n,\tau} = \frac{R_{n,\tau}}{P_{n,\tau}} - \delta. \tag{80}$$

The corresponding gross real payoff is $1 + r_{n,\tau}$. For an unconstrained household born at date t,

$$u'\left(c_{n,g,t+g-1}\right) = \beta \psi_{n,t+g} s_{n,g+1,t+g} \left(1 + r_{n,t+g}\right) u'\left(c_{n,g+1,t+g}\right), \qquad g = 1, \ldots, G-1. \tag{81}$$

Under CRRA preferences,

$$\frac{c_{n,g+1,t+g}}{c_{n,g,t+g-1}} = \left[\beta \psi_{n,t+g} s_{n,g+1,t+g} \left(1 + r_{n,t+g}\right)\right]^{\rho}. \tag{82}$$

Accidental bequests and transfers Date-t capital was purchased by households alive at $t-1$:

$$K_{n,t} = \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}. \tag{83}$$

Only $N_{n,g,t}$ households survive to age g. Claims left by households that do not survive are

$$K_{n,t}^{B} \equiv \sum_{g=2}^{G} \left(N_{n,g-1,t-1} - N_{n,g,t}\right) a_{n,g,t}. \tag{84}$$

Their date-t payoff is

$$T_{n,t}^{B} = P_{n,t} \left(1 + r_{n,t}\right) K_{n,t}^{B} = \left[R_{n,t} + (1-\delta) P_{n,t}\right] K_{n,t}^{B}. \tag{85}$$

Under the baseline equal-per-capita redistribution rule,

$$tr_{n,t}^{D} = \frac{T_{n,t}^{B}}{N_{n,t}}, \qquad \sum_{g=1}^{G} N_{n,g,t} tr_{n,t}^{D} = T_{n,t}^{B}. \tag{86}$$

The international transfer is

$$tr^T_{n,t} = \frac{D_{n,t}}{N_{n,t}}, \qquad \sum_{n=1}^{N} D_{n,t} = 0. \tag{87}$$

This closure preserves international capital-market autarky. Endogenous international asset trade would replace $D_{n,t}$ with current-account and asset-market conditions.

Aggregation and capital accumulation Aggregate consumption and physical capital are

$$C_{n,t} = \sum_{g=1}^{G} N_{n,g,t} c_{n,g,t}, \tag{88}$$

$$K_{n,t} = \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}, \tag{89}$$

$$K_{n,t+1} = \sum_{g=1}^{G-1} N_{n,g,t} a_{n,g+1,t+1}. \tag{90}$$

Investment is

$$I_{n,t} = K_{n,t+1} - (1-\delta) K_{n,t}. \tag{91}$$

A.1.3 Composite demand and production cost

The expenditure-minimization problem associated with (9) gives

$$P_{n,t} = \left[\int_0^1 p_{n,t}(\omega)^{1-\sigma} \, \mathrm{d}\omega \right]^{\frac{1}{1-\sigma}}, \tag{92}$$

$$x_{n,t}(\omega) = \left(\frac{p_{n,t}(\omega)}{P_{n,t}} \right)^{-\sigma} Q_{n,t}. \tag{93}$$

Variety expenditure is

$$p_{n,t}(\omega) x_{n,t}(\omega) = X_{n,t} \left(\frac{p_{n,t}(\omega)}{P_{n,t}} \right)^{1-\sigma}. \tag{94}$$

For productivity normalized to one, a producer solves

$$\min_{h,k,m \geq 0} \{ W_{i,t} h + R_{i,t} k + P_{i,t} m \} \quad \text{s.t.} \quad (h^{\gamma_{L,i}} k^{1-\gamma_{L,i}})^{\gamma_i} m^{1-\gamma_i} \geq 1. \tag{95}$$

Define

$$\Upsilon_i \equiv (\gamma_i \gamma_{L,i})^{-\gamma_i \gamma_{L,i}} [\gamma_i (1-\gamma_{L,i})]^{-\gamma_i (1-\gamma_{L,i})} (1-\gamma_i)^{-(1-\gamma_i)}. \tag{96}$$

The unit input cost is

$$c_{i,t} = \Upsilon_i \left(W_{i,t}^{\gamma_{L,i}} R_{i,t}^{1-\gamma_{L,i}} \right)^{\gamma_i} P_{i,t}^{1-\gamma_i}. \tag{97}$$

A point with productivity q delivers to destination n at cost

$$c_{ni,t}(q) = \frac{\kappa_{ni,t} c_{i,t}}{q}. \tag{98}$$

A.1.4 Pareto–Poisson foundation of the knowledge frontier

Introduce a temporary quality lower bound $\bar{z} > 0$. Conditional idea quality has Pareto CDF

$$H_{\bar{z}}(z) = \begin{cases} 0, & z < \bar{z}, \\ 1 - (z/\bar{z})^{-\theta}, & z \geq \bar{z}. \end{cases} \tag{99}$$

For each variety, the number of new ideas generated during t is

$$\mathcal{N}_{i,t}^{\text{new},\bar{z}}(\omega) \sim \text{Poisson}\left(\Delta\lambda_{i,t} \bar{z}^{-\theta} \right). \tag{100}$$

Conditional on the count, quality draws are independent and distributed according to $H_{\bar{z}}$.

For $q \geq \bar{z}$, the probability that all new ideas lie below q is

$$\begin{aligned} F_{i,t}^{\text{best,new}}(q;\bar{z}) &= \exp\left\{ -\Delta\lambda_{i,t} \bar{z}^{-\theta} [1 - H_{\bar{z}}(q)] \right\} \\ &= \exp[-\Delta\lambda_{i,t} q^{-\theta}], \end{aligned} \tag{101}$$

because $1 - H_{\bar{z}}(q) = q^{-\theta} \bar{z}^{\theta}$. Taking $\bar{z} \downarrow 0$ gives

$$F_{i,t}^{\text{best,new}}(q) = \exp[-\Delta\lambda_{i,t} q^{-\theta}]. \tag{102}$$

Poisson thinning gives the number of new ideas above any fixed $q > 0$:

$$\mathcal{N}_{i,t}^{\text{new}}(\omega; q) \sim \text{Poisson}(\Delta\lambda_{i,t} q^{-\theta}). \tag{103}$$

The inherited threshold count is

$$\mathcal{N}_{i,t}(\omega; q) \sim \text{Poisson}(\lambda_{i,t} q^{-\theta}). \tag{104}$$

Independent inherited and new processes superimpose, so

$$\lambda_{i,t+1} = \lambda_{i,t} + \Delta\lambda_{i,t}. \tag{105}$$

The best point in the accumulated process therefore satisfies

$$F_{i,t}(q) = \exp[-\lambda_{i,t} q^{-\theta}]. \tag{106}$$

A.1.5 Delivered-cost process, pricing, and trade shares

A productivity point delivers cost below c if $q \geq c_{i,t}\kappa_{ni,t}/c$. Hence the expected number of origin-i cost points below c is

$$\Psi_{ni,t}(c) = \lambda_{i,t}(c_{i,t}\kappa_{ni,t})^{-\theta} c^{\theta}. \tag{107}$$

Let

$$\varphi_{ni,t} \equiv \lambda_{i,t}(c_{i,t}\kappa_{ni,t})^{-\theta}, \qquad \Phi_{n,t} \equiv \sum_{\ell} \varphi_{n\ell,t}. \tag{108}$$

The aggregate cost process has mean measure $\Phi_{n,t}c^{\theta}$.

Let $C_{1,n,t}(\omega)$ and $C_{2,n,t}(\omega)$ denote the lowest and second-lowest delivered costs. Transform

$$Y_k = \Phi_{n,t} C_{k,n,t}(\omega)^{\theta}, \qquad k = 1, 2. \tag{109}$$

The transformed points form a unit-rate Poisson process, so

$$Y_1 \sim \mathrm{Exp}(1), \qquad Y_2 - Y_1 \sim \mathrm{Exp}(1), \tag{110}$$

with independent spacings. Let $U = Y_1/Y_2$ and $V = Y_2$. Their joint density factorizes:

$$f_{U,V}(u, v) = v e^{-v}, \qquad 0 < u < 1, \quad v > 0. \tag{111}$$

Thus

$$U \sim \mathrm{Uniform}(0, 1), \qquad V \sim \mathrm{Gamma}(2, 1), \qquad U \perp V, \tag{112}$$

and

$$C_{2,n,t} = \left(\frac{V}{\Phi_{n,t}}\right)^{1/\theta}, \qquad \frac{C_{1,n,t}}{C_{2,n,t}} = U^{1/\theta}. \tag{113}$$

Under the Bertrand rule,

$$p_{n,t}(\omega) = C_{2,n,t}(\omega). \tag{114}$$

An origin-i point at cost c wins if no other point lies below it:

$$\begin{aligned} \Pr(i \text{ supplies } n) &= \int_0^{\infty} \theta \varphi_{ni,t} c^{\theta-1} e^{-\Phi_{n,t}c^{\theta}}\, \mathrm{d}c \\ &= \frac{\varphi_{ni,t}}{\Phi_{n,t}}. \end{aligned} \tag{115}$$

The origin mark is independent of V, while price depends on V. Supplier probability therefore equals the CES expenditure share:

$$\pi_{ni,t} = \frac{\lambda_{i,t}(c_{i,t}\kappa_{ni,t})^{-\theta}}{\sum_\ell \lambda_{\ell,t}(c_{\ell,t}\kappa_{n\ell,t})^{-\theta}}. \tag{116}$$

Since $V \sim \text{Gamma}(2,1)$,

$$\begin{aligned} P_{n,t}^{1-\sigma} &= \mathbb{E}[p_{n,t}(\omega)^{1-\sigma}] \\ &= \Phi_{n,t}^{\frac{\sigma-1}{\theta}} \Gamma\left(2 - \frac{\sigma-1}{\theta}\right), \end{aligned} \tag{117}$$

which gives

$$P_{n,t} = \Gamma\left(2 - \frac{\sigma-1}{\theta}\right)^{\frac{1}{1-\sigma}} \Phi_{n,t}^{-1/\theta}. \tag{118}$$

The winner's variable-cost share of revenue is

$$\mathbb{E}\left[\frac{C_1}{C_2}\right] = \mathbb{E}[U^{1/\theta}] = \frac{\theta}{1+\theta}. \tag{119}$$

Consequently,

$$\Pi_{i,t}^{\text{op}} = \frac{1}{1+\theta} Y_{i,t}. \tag{120}$$

A.1.6 Factor payments and static equilibrium

Factor and intermediate market clearing imply

$$W_{i,t}\mathcal{L}_{i,t}^P = \gamma_i \gamma_{L,i} \frac{\theta}{1+\theta} Y_{i,t}, \tag{121}$$

$$R_{i,t}K_{i,t} = \gamma_i(1-\gamma_{L,i}) \frac{\theta}{1+\theta} Y_{i,t}, \tag{122}$$

$$P_{i,t}M_{i,t} = (1-\gamma_i) \frac{\theta}{1+\theta} Y_{i,t}. \tag{123}$$

Thus

$$R_{i,t} = \frac{1-\gamma_{L,i}}{\gamma_{L,i}} \frac{W_{i,t}\mathcal{L}_{i,t}^P}{K_{i,t}}, \tag{124}$$

$$M_{i,t} = \frac{1-\gamma_i}{\gamma_i \gamma_{L,i}} \frac{W_{i,t}\mathcal{L}_{i,t}^P}{P_{i,t}}, \tag{125}$$

$$Y_{i,t} = \frac{1+\theta}{\theta \gamma_i \gamma_{L,i}} W_{i,t}\mathcal{L}_{i,t}^P. \tag{126}$$

Since $X_{n,t} = Y_{n,t} + D_{n,t}$, the relative-wage system is

$$\frac{W_{i,t}\mathcal{L}^P_{i,t}}{\gamma_i \gamma_{L,i}} = \sum_n \pi_{ni,t} \left[\frac{W_{n,t}\mathcal{L}^P_{n,t}}{\gamma_n \gamma_{L,n}} + \frac{\theta}{1+\theta} D_{n,t} \right]. \tag{127}$$

Equations (97), (116), (118), (124), and (127) form the static fixed point given $(K_{i,t}, \lambda_{i,t}, \mathcal{L}^P_{i,t})_i$. One nominal variable is normalized each period.

A.1.7 Innovation firms and the research cutoff

Let $j \in [0,1]$ index atomistic innovation firms. Let $L_{i,g,t}(j)$ be firm j's active age-g workforce, with

$$\int_0^1 L_{i,g,t}(j)\,\mathrm{d}j = L_{i,g,t}. \tag{128}$$

Firm j chooses $\tilde{s}^R_{i,g,t}(j) \in [0,1]$. Its capitalized surplus from this age group is

$$\begin{aligned} \mathcal{S}^R_{i,g,t}(j, \tilde{s}^R) &= L_{i,g,t}(j) \int_0^{\tilde{s}^R} \left[v_{i,t} b_{i,g,t}(u) - W_{i,t} l_{i,g} \right] \mathrm{d}u \\ &= L_{i,g,t}(j) \left[v_{i,t} \alpha_{i,t} \eta_g \left(\tilde{s}^R \right)^{\xi} - W_{i,t} l_{i,g} \tilde{s}^R \right]. \end{aligned} \tag{129}$$

The firm is negligible, so

$$\frac{\partial v_{i,t}}{\partial \tilde{s}^R_{i,g,t}(j)} = 0, \qquad \frac{\partial W_{i,t}}{\partial \tilde{s}^R_{i,g,t}(j)} = 0 \tag{130}$$

in its private problem. The interior first-order condition is

$$\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g [\tilde{s}^{R,*}_{i,g,t}(j)]^{\xi-1} = W_{i,t} l_{i,g}. \tag{131}$$

Strict concavity follows from $0 < \xi < 1$.

A.1.8 Why participation does not enter the private cutoff

Using $L_{i,g,t}(j) = (1 - \tau^L_{i,t}) N_{i,g,t}(j)$, the derivative of (129) is

$$\frac{\partial \mathcal{S}^R_{i,g,t}}{\partial \tilde{s}^R} = (1 - \tau^L_{i,t}) N_{i,g,t}(j) \left[\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g \left(\tilde{s}^R \right)^{\xi-1} - W_{i,t} l_{i,g} \right]. \tag{132}$$

For $1 - \tau^L_{i,t} > 0$, the multiplicative participation term cancels from the first-order condition. Therefore

$$\left. \frac{\partial s^R_{i,g,t}}{\partial \tau^L_{i,t}} \right|_{v_{i,t}, W_{i,t}, \lambda_{i,t}} = 0. \tag{133}$$

This is a private, partial-equilibrium statement. In the full equilibrium, participation changes labor supplies, knowledge accumulation, wages, profits, capital, and the value of knowledge, so generally

$$\frac{\mathrm{d}s^*_{i,g,t}}{\mathrm{d}\tau^L_{i,t}} \neq 0. \tag{134}$$

Symmetric firms choose the same cutoff. Aggregate age-group innovation is

$$\Delta\lambda_{i,g,t} = \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g \left(s^R_{i,g,t}\right)^{\xi} L_{i,g,t}. \tag{135}$$

For interior age groups,

$$s^R_{i,g,t} = \left(\frac{\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g}{W_{i,t} l_{i,g}}\right)^{\frac{1}{1-\xi}}. \tag{136}$$

The upper-corner condition is

$$\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t} \eta_g - W_{i,t} l_{i,g} \geq 0. \tag{137}$$

For interior age groups, define

$$A^R_{i,t} \equiv \left(\frac{\xi v_{i,t} \bar{\alpha}_i \lambda^{\nu}_{i,t}}{W_{i,t}}\right)^{\frac{1}{1-\xi}}. \tag{138}$$

Then

$$s^R_{i,g,t} = A^R_{i,t} \left(\frac{\eta_g}{l_{i,g}}\right)^{\frac{1}{1-\xi}}. \tag{139}$$

The share of age group g in national researcher headcount is

$$\frac{L^R_{i,g,t}}{\sum_h L^R_{i,h,t}} = \frac{L_{i,g,t}(\eta_g / l_{i,g})^{1/(1-\xi)}}{\sum_h L_{i,h,t}(\eta_h / l_{i,h})^{1/(1-\xi)}}. \tag{140}$$

If participation is common across ages, the common factor $1 - \tau^L_{i,t}$ cancels from researcher composition, although it changes the total number of researchers.

A.1.9 Two corporate activities and consolidated payouts

The model can be interpreted using two legally separate firms or two divisions of an integrated firm.

Let $\mathcal{T}^K_{i,t}$ be the royalty or licensing payment from variety producers to innovation firms.

Production-firm and innovation-firm payouts are

$$D^P_{i,t} = \Pi^{\mathrm{op}}_{i,t} - \mathcal{T}^K_{i,t}, \tag{141}$$

$$D^R_{i,t} = \mathcal{T}^K_{i,t} - W_{i,t}\mathcal{L}^R_{i,t}. \tag{142}$$

Their consolidated payout is

$$D^F_{i,t} \equiv D^P_{i,t} + D^R_{i,t} = \Pi^{\mathrm{op}}_{i,t} - W_{i,t}\mathcal{L}^R_{i,t}. \tag{143}$$

The internal royalty cancels. Therefore the quantitative model does not need to determine $\mathcal{T}^K_{i,t}$ separately. Domestic households own both activities and receive

$$d^F_{i,t} = \frac{D^F_{i,t}}{N_{i,t}}. \tag{144}$$

A negative payout represents a household equity injection used to finance the current research wage bill.

A.1.10 Knowledge cohorts and the value of knowledge

Treat $\lambda_{i,0}$ as the stock of knowledge inherited at the beginning of the model. Research conducted during period r creates a new knowledge cohort of size $\Delta\lambda_{i,r}$. Because knowledge does not depreciate, the stock available in country i at a future date τ is

$$\lambda_{i,\tau} = \lambda_{i,0} + \sum_{r=0}^{\tau-1} \Delta\lambda_{i,r}. \tag{145}$$

Each knowledge cohort consists of productivity points defined over varieties ω and productivity levels z. We assume that all cohorts draw from the same variety–productivity distribution, with common kernel

$$\mathrm{d}\omega\, \theta z^{-\theta-1}\, \mathrm{d}z. \tag{146}$$

Cohorts therefore differ in their mass, but not in the distribution of the varieties to which their ideas apply or in the distribution of idea quality.

More formally, the knowledge points created during period r form an independent Poisson point process with mean measure

$$\nu_{i,r}(\mathrm{d}\omega, \mathrm{d}z) = \Delta\lambda_{i,r}\, \mathrm{d}\omega\, \theta z^{-\theta-1}\, \mathrm{d}z. \tag{147}$$

The inherited stock has the same kernel with scale $\lambda_{i,0}$. Because independent Poisson processes add, the aggregate stock at date τ has mean measure

$$\nu_{i,\tau}(\mathrm{d}\omega, \mathrm{d}z) = \lambda_{i,\tau}\,\mathrm{d}\omega\,\theta z^{-\theta-1}\,\mathrm{d}z. \tag{148}$$

The economic implication is straightforward. Consider any finite region of the variety–productivity space. The expected number of knowledge points in that region created during period t is proportional to $\Delta\lambda_{i,t}$, while the expected total number of points is proportional to $\lambda_{i,\tau}$. Hence, the expected share of the date-t cohort in the date-τ knowledge stock is

$$\Pr(\text{knowledge point belongs to the date-}t\text{ cohort}) = \frac{\Delta\lambda_{i,t}}{\lambda_{i,\tau}}, \qquad t < \tau. \tag{149}$$

Thus, a cohort that accounts for a larger fraction of the accumulated knowledge stock is proportionally more likely to supply the technology used in future production.

Market selection and operating profits depend on a knowledge point's variety, productivity, and the productivity of competing points. They do not depend on when the point was created, the age of its inventor, or the identity of the R&D division that created it. It follows that the expected share of date-τ operating profits attributable to the date-t knowledge cohort is the same as its share of the knowledge stock:

$$\Pi_{i,\tau}^{(t)} = \frac{\Delta\lambda_{i,t}}{\lambda_{i,\tau}}\Pi_{i,\tau}^{\mathrm{op}}. \tag{150}$$

Dividing by the size of the cohort gives the expected date-τ operating profit generated by one unit of the Poisson knowledge scale:

$$\frac{\Pi_{i,\tau}^{(t)}}{\Delta\lambda_{i,t}} = \frac{\Pi_{i,\tau}^{\mathrm{op}}}{\lambda_{i,\tau}}. \tag{151}$$

This object is an average expected payoff per unit of knowledge intensity. It is not the derivative of aggregate operating profits with respect to the aggregate knowledge stock.

A nominal unit invested in physical capital at t earns the nominal gross return

$$\mathcal{R}_{i,t+1}^{N} = \frac{P_{i,t+1}}{P_{i,t}}(1 + r_{i,t+1}). \tag{152}$$

The date-t price of one nominal unit delivered at $\tau > t$ is

$$m_{i,t,\tau}^{K} = \frac{P_{i,t}}{P_{i,\tau}}\prod_{u=t+1}^{\tau}(1 + r_{i,u})^{-1}. \tag{153}$$

No-arbitrage implies

$$v_{i,t} = \sum_{\tau=t+1}^{\infty} m^K_{i,t,\tau} \frac{\Pi^{\mathrm{op}}_{i,\tau}}{\lambda_{i,\tau}}. \tag{154}$$

Separating the first payoff gives

$$v_{i,t} = \frac{P_{i,t}}{P_{i,t+1}(1+r_{i,t+1})} \left[\frac{\Pi^{\mathrm{op}}_{i,t+1}}{\lambda_{i,t+1}} + v_{i,t+1} \right]. \tag{155}$$

The no-bubble condition is

$$\lim_{T\to\infty} m^K_{i,t,T} v_{i,T} = 0. \tag{156}$$

For an interior age group, multiplying (131) by $s^R_{i,g,t} L_{i,g,t}$ gives

$$W_{i,t} l_{i,g} s^R_{i,g,t} L_{i,g,t} = \xi v_{i,t} \Delta\lambda_{i,g,t}. \tag{157}$$

The left side is the age-group research wage bill. The equality allocates the capitalized value of future profits; it is not a current resource identity.

A.1.11 Aggregate household accounting

Now, aggregate individual variables across age groups:

$$C_{n,t} \equiv \sum_{g=1}^{G} N_{n,g,t} c_{n,g,t}, \qquad K_{n,t} \equiv \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}. \tag{D.1}$$

The individual budget constraint in the single-sector model is

$$P_{n,t} c_{n,g,t} + P_{n,t} a_{n,g+1,t+1} = P_{n,t}(1+r_{n,t}) a_{n,g,t} + W_{n,t}(1-\tau^L_{n,t}) l_{n,g} + d^F_{n,t} + tr^D_{n,t} + tr^T_{n,t}.$$

Here $d^F_{n,t} = D^F_{n,t}/N_{n,t}$ is the per-capita corporate payout, $tr^D_{n,t}$ is the accidental-bequest transfer, and $tr^T_{n,t} = D_{n,t}/N_{n,t}$ is the per-capita trade transfer. Accidental bequests satisfy

$$tr^D_{n,t} \equiv P_{n,t}(1+r_{n,t}) \frac{\sum_{g=2}^{G} \left(N_{n,g-1,t-1} - N_{n,g,t}\right) a_{n,g,t}}{N_{n,t}}. \tag{D.2}$$

Aggregating individual budget constraints across ages gives

$$P_{n,t}\sum_{g=1}^{G}N_{n,g,t}c_{n,g,t}+P_{n,t}\sum_{g=1}^{G}N_{n,g,t}a_{n,g+1,t+1}$$

$$=P_{n,t}(1+r_{n,t})\sum_{g=1}^{G}N_{n,g,t}a_{n,g,t} \tag{D.3}$$

$$+\sum_{g=1}^{G}N_{n,g,t}W_{n,t}(1-\tau_{n,t}^{L})l_{n,g}+\sum_{g=1}^{G}N_{n,g,t}d_{n,t}^{F}$$

$$+\sum_{g=1}^{G}N_{n,g,t}tr_{n,t}^{D}+\sum_{g=1}^{G}N_{n,g,t}tr_{n,t}^{T}. \tag{D.4}$$

Using the definitions of aggregate consumption, capital, accidental bequests, corporate payouts, and trade transfers,

$$P_{n,t}C_{n,t}+P_{n,t}K_{n,t+1}=W_{n,t}\left(\mathcal{L}_{n,t}^{P}+\mathcal{L}_{n,t}^{R}\right)+D_{n,t}^{F}$$

$$+(1+r_{n,t})P_{n,t}\sum_{g=1}^{G}N_{n,g,t}a_{n,g,t} \tag{D.5}$$

$$+(1+r_{n,t})P_{n,t}\sum_{g=2}^{G}\left(N_{n,g-1,t-1}-N_{n,g,t}\right)a_{n,g,t}+D_{n,t} \tag{D.6}$$

$$=(1+r_{n,t})P_{n,t}K_{n,t}+W_{n,t}\left(\mathcal{L}_{n,t}^{P}+\mathcal{L}_{n,t}^{R}\right)+D_{n,t}^{F}+D_{n,t} \tag{D.7}$$

$$=\left(1+\frac{R_{n,t}}{P_{n,t}}-\delta\right)P_{n,t}K_{n,t}+W_{n,t}\left(\mathcal{L}_{n,t}^{P}+\mathcal{L}_{n,t}^{R}\right)$$

$$+D_{n,t}^{F}+D_{n,t}. \tag{D.8}$$

Aggregate investment is

$$I_{n,t}\equiv K_{n,t+1}-(1-\delta)K_{n,t}. \tag{D.9}$$

Therefore,

$$P_{n,t}C_{n,t}+P_{n,t}I_{n,t}=(r_{n,t}+\delta)P_{n,t}K_{n,t}+W_{n,t}\left(\mathcal{L}_{n,t}^{P}+\mathcal{L}_{n,t}^{R}\right)+D_{n,t}^{F}+D_{n,t} \tag{D.10}$$

$$=R_{n,t}K_{n,t}+W_{n,t}\left(\mathcal{L}_{n,t}^{P}+\mathcal{L}_{n,t}^{R}\right)+D_{n,t}^{F}+D_{n,t}$$

$$=R_{n,t}K_{n,t}+W_{n,t}\mathcal{L}_{n,t}^{P}+\Pi_{n,t}^{\mathrm{op}}+D_{n,t}, \tag{D.11}$$

where the final equality uses $D_{n,t}^{F}=\Pi_{n,t}^{\mathrm{op}}-W_{n,t}\mathcal{L}_{n,t}^{R}$.

Summing individual budgets gives

$$
\begin{aligned}
P_{i,t}C_{i,t} + P_{i,t}K_{i,t+1} = {} & \left[R_{i,t} + (1-\delta)P_{i,t}\right] K_{i,t} \\
& + W_{i,t}(1-\tau^{L}_{i,t}) \sum_{g} l_{i,g} N_{i,g,t} + D^{F}_{i,t} + D_{i,t},
\end{aligned} \tag{158}
$$

where accidental bequests cancel through (86). Since

$$
(1-\tau^{L}_{i,t}) \sum_{g} l_{i,g} N_{i,g,t} = \mathcal{L}^{P}_{i,t} + \mathcal{L}^{R}_{i,t}, \tag{159}
$$

subtracting undepreciated capital yields

$$
\begin{aligned}
P_{i,t}C_{i,t} + P_{i,t}I_{i,t} = {} & W_{i,t}\mathcal{L}^{P}_{i,t} + W_{i,t}\mathcal{L}^{R}_{i,t} + R_{i,t}K_{i,t} \\
& + D^{F}_{i,t} + D_{i,t}.
\end{aligned} \tag{160}
$$

Using $D^{F}_{i,t} = \Pi^{\mathrm{op}}_{i,t} - W_{i,t}\mathcal{L}^{R}_{i,t}$ gives

$$
P_{i,t}C_{i,t} + P_{i,t}I_{i,t} = W_{i,t}\mathcal{L}^{P}_{i,t} + R_{i,t}K_{i,t} + \Pi^{\mathrm{op}}_{i,t} + D_{i,t}. \tag{161}
$$

Research wages are visible in (160); they cancel only after household and corporate accounts are consolidated.

Production accounting is

$$
Y_{i,t} = W_{i,t}\mathcal{L}^{P}_{i,t} + R_{i,t}K_{i,t} + P_{i,t}M_{i,t} + \Pi^{\mathrm{op}}_{i,t}. \tag{162}
$$

Combining (161) and (162) gives

$$
P_{i,t}(C_{i,t} + I_{i,t} + M_{i,t}) = Y_{i,t} + D_{i,t} = X_{i,t}, \tag{163}
$$

which is equivalent to $Q_{i,t} = C_{i,t} + I_{i,t} + M_{i,t}$.

A.1.12 Complete transitional equilibrium

Given initial conditions

$$
\left\{ \lambda_{i,0}, \{a_{n,g,0}\}_{g=1}^{G} \right\}_{i=1}^{N}, \tag{164}
$$

exogenous demographic and participation paths, trade costs, trade deficits, preference shifters, and parameters, a transitional equilibrium consists of sequences

$$\left\{\begin{array}{l} N_{i,g,t}, L_{i,g,t}, L^P_{i,g,t}, L^R_{i,g,t}, s^R_{i,g,t}, \\ c_{i,g,t}, a_{i,g+1,t+1}, C_{i,t}, I_{i,t}, K_{i,t+1}, \\ \Delta\lambda_{i,g,t}, \lambda_{i,t+1}, v_{i,t}, \\ W_{i,t}, R_{i,t}, P_{i,t}, c_{i,t}, M_{i,t}, \\ X_{i,t}, Y_{i,t}, \Pi^{\mathrm{op}}_{i,t}, D^F_{i,t}, \\ \pi_{ni,t}, X_{ni,t}, tr^D_{i,g,t}, tr^T_{i,t} \end{array}\right\}_{i,n,g,t} \tag{165}$$

satisfying the following conditions.

Demography and participation. Equations (1)–(3) and (69) hold.

Labor allocation. Equations (70)–(71) hold.

Household optimality. Households solve (73) subject to (74) and (75), including the Euler equation or borrowing-constraint inequality.

Capital accumulation. Equations (90) and (91) hold.

Production and trade. Equations (97), (116), (118), (120), and (121)–(123) hold, together with $X_{ni,t} = \pi_{ni,t} X_{n,t}$.

Innovation. Equations (135), (105), and the research cutoff (136) or its Kuhn–Tucker counterpart hold.

Corporate valuation and payouts. Equations (143), (144), and (155) hold.

Transfers and clearing. Equations (86), (87), and (163) hold.

Normalization and terminal conditions. One nominal variable is normalized each date, and (156) holds. A finite-horizon computation uses terminal conditions from the terminal balanced growth path.

A.1.13 Balanced growth and stationary equilibrium

Definition Assume that, after date T, all countries share a constant population growth rate g_n, stable age shares, constant survival and fertility rates, constant participation, constant country–age efficiency units $l_{n,g}$, constant trade costs, constant saving wedges, and constant parameters. Assume further that research-assignment cutoffs are interior and stationary and that the gross-output labor share

$$\alpha_L \equiv \gamma_i \gamma_{L,i} \tag{166}$$

is common across countries.

A balanced growth path is a transitional equilibrium for which

$$n_{n,g,t}, \quad s^R_{n,g,t}, \quad \pi_{ni,t}, \quad r_{n,t}, \quad \text{and all age profiles after removing their common growth trend} \tag{167}$$

are constant.

Knowledge growth Under stable demographics and participation,

$$L_{n,g,t} = (1+g_n)^{t-T} L_{n,g,T}, \qquad t \geq T. \tag{168}$$

Because the age distribution and the research-assignment shares are stationary,

$$\frac{\sum_g \eta_g (s^R_{n,g,t})^{\xi} L_{n,g,t}}{L_{n,t}} = \frac{\sum_g \eta_g (s^R_{n,g,T})^{\xi} L_{n,g,T}}{L_{n,T}}, \qquad t \geq T. \tag{169}$$

The knowledge law remains

$$\lambda_{n,t+1} = \lambda_{n,t} + \bar{\alpha}_n \lambda^{\nu}_{n,t} \sum_g \eta_g (s^R_{n,g,t})^{\xi} L_{n,g,t}. \tag{170}$$

A constant knowledge growth rate requires $L_{n,t}/\lambda^{1-\nu}_{n,t}$ to be stationary. Therefore,

$$(1+g_\lambda)^{1-\nu} = 1+g_n, \qquad 1+g_\lambda = (1+g_n)^{1/(1-\nu)}. \tag{171}$$

A fixed quantile of the Fréchet frontier is proportional to $\lambda^{1/\theta}_{n,t}$, so

$$1+g_A = (1+g_\lambda)^{1/\theta}. \tag{172}$$

The ratio

$$\frac{\lambda_{n,t}}{L^{1/(1-\nu)}_{n,t}} \tag{173}$$

is stationary. Its balanced-growth level is

$$\frac{\lambda_{n,t}}{L_{n,t}^{1/(1-\nu)}} = \left[\frac{\bar{\alpha}_n}{g_\lambda} \frac{\sum_g \eta_g (s_{n,g,T}^R)^\xi L_{n,g,T}}{L_{n,T}}\right]^{1/(1-\nu)}, \qquad t \geq T. \tag{174}$$

Per-capita and aggregate growth Let g_ω be the growth rate of real per-capita quantities and let g_K be the common growth rate of aggregate capital, intermediates, output, and real expenditure. Along a balanced path,

$$1 + g_K = (1 + g_n)(1 + g_\omega). \tag{175}$$

Production efficiency labor grows at $1 + g_n$. Applying these growth rates to (10),

$$\begin{aligned} 1 + g_K &= (1 + g_A) \left[(1 + g_n)^{\gamma_{L,i}} (1 + g_K)^{1-\gamma_{L,i}}\right]^{\gamma_i} (1 + g_K)^{1-\gamma_i} \\ &= (1 + g_A)(1 + g_n)^{\alpha_L} (1 + g_K)^{1-\alpha_L}. \end{aligned} \tag{176}$$

Hence,

$$1 + g_\omega = (1 + g_A)^{1/\alpha_L} = (1 + g_n)^{1/[\theta\alpha_L(1-\nu)]}. \tag{177}$$

Individual consumption, individual assets, the real wage per efficiency unit, per-capita payouts, and per-capita transfers grow at $1 + g_\omega$. Aggregate consumption, capital, investment, sales, output, real expenditure, and intermediates grow at $1 + g_K$. The net real capital return is stationary.

Real unit input cost is

$$\frac{c_{n,t}}{P_{n,t}} = \Upsilon_n \left[\left(\frac{W_{n,t}}{P_{n,t}}\right)^{\gamma_{L,n}} \left(\frac{R_{n,t}}{P_{n,t}}\right)^{1-\gamma_{L,n}}\right]^{\gamma_n}. \tag{178}$$

Its growth rate satisfies

$$\frac{c_{n,t+1}/P_{n,t+1}}{c_{n,t}/P_{n,t}} = (1 + g_\omega)^{\alpha_L} = 1 + g_A. \tag{179}$$

Therefore,

$$(1 + g_\lambda)(1 + g_A)^{-\theta} = 1, \tag{180}$$

and bilateral trade shares are stationary.

Stationary household profiles and the capital level For $t \geq T$, the stationary age profiles satisfy

$$c_{n,g,t+1} = (1 + g_\omega)c_{n,g,t}, \qquad a_{n,g,t+1} = (1 + g_\omega)a_{n,g,t}. \tag{181}$$

For an unconstrained household, evaluate the Euler equation at calendar date T. Since $c_{n,g+1,T+1} = (1+g_\omega)c_{n,g+1,T}$, we obtain

$$(1+g_\omega)\frac{c_{n,g+1,T}}{c_{n,g,T}} = \left[\beta\psi_n s_{n,g+1,T+1}(1+r_{n,T+1})\right]^\rho, \qquad g = 1,\ldots,G-1. \tag{182}$$

The age-specific budgets at date T, the asset boundaries, transfer clearing, and capital aggregation determine $\{c_{n,g,T}, a_{n,g,T}\}_{g=1}^G$ and the balanced-growth capital level.

Stationary research incentives Real operating profit grows at $1+g_K$, while knowledge grows at $1+g_\lambda$. Therefore,

$$\frac{\Pi^{\text{op}}_{n,t+1}/(P_{n,t+1}\lambda_{n,t+1})}{\Pi^{\text{op}}_{n,t}/(P_{n,t}\lambda_{n,t})} = \frac{(1+g_n)(1+g_\omega)}{1+g_\lambda} = \frac{1+g_\omega}{(1+g_\lambda)^\nu}. \tag{183}$$

If

$$\frac{(1+g_\omega)/(1+g_\lambda)^\nu}{1+r_{n,T}} < 1, \tag{184}$$

the no-bubble value recursion implies

$$\frac{v_{n,t+1}/P_{n,t+1}}{v_{n,t}/P_{n,t}} = \frac{1+g_\omega}{(1+g_\lambda)^\nu}. \tag{185}$$

It follows that

$$\frac{(v_{n,t}/P_{n,t})\lambda^\nu_{n,t}}{W_{n,t}/P_{n,t}} \tag{186}$$

is stationary, which supports constant interior research-assignment shares.

Stationary steady state If $g_n = 0$, then

$$g_\lambda = g_A = g_\omega = g_K = 0. \tag{187}$$

All undetrended quantities are stationary. The steady state is the special case of the balanced growth path with zero population growth and no other exogenous trend growth.

A.1.14 Equilibrium conditions

Table 2: Stationary balanced-growth equilibrium

$1+g_n$	$Z_{n,g,t+1} = (1+g_n)Z_{n,g,t}$; $\quad Z \in \left\{N_{n,g}, N_n, L_{n,g}, L^P_{n,g}, L^R_{n,g}, L^e_n, \mathcal{L}^P_n, \mathcal{L}^R_n\right\}$; $\quad n_{n,g,t} \equiv \frac{N_{n,g,t}}{N_{n,t}}$; $\quad \ell_{n,g,t+1} = n_{n,g,t}$	$\forall n, g,\ t \in [T,\infty)$
$(1+g_\lambda)$	$\lambda_{n,t+1} = (1+g_\lambda)\lambda_{n,t}$; $\quad (1+g_\lambda) = (1+g_n)^{\frac{1}{1-\nu}}$; $\quad (1+g_A) \equiv (1+g_\lambda)^{1/\theta}$	$\forall n,\ t \in [T,\infty)$
$(1+g_\omega)$	$Z_{n,g,t+1} = (1+g_\omega)Z_{n,g,t}$; $\quad Z \in \left\{c_{n,g}, a_{n,g}, \frac{W_n}{P_n}, \frac{d^F_n}{P_n}, \frac{tr^D_n}{P_n}, \frac{tr^T_n}{P_n}\right\}$; $\quad (1+g_\omega) = (1+g_A)^{1/\alpha_L} = (1+g_n)^{\frac{1}{\theta\alpha_L(1-\nu)}}$	$\forall n, g,\ t \in [T,\infty)$
$(1+g_K)$	$Z_{n,t+1} = (1+g_K)Z_{n,t}$; $\quad Z \in \left\{C_n, I_n, K_n, M_n, Q_n, \frac{X_n}{P_n}, \frac{Y_n}{P_n}, \frac{D_n}{P_n}, \frac{\Pi^{\mathrm{op}}_n}{P_n}, \frac{D^F_n}{P_n}\right\}$; $\quad (1+g_K) = (1+g_n)(1+g_\omega)$	$\forall n,\ t \in [T,\infty)$
$1+g_{rc_n}$	$\frac{c_{n,t+1}}{P_{n,t+1}} = (1+g_A)\frac{c_{n,t}}{P_{n,t}}$; $\quad (1+g_A) = (1+g_\omega)^{\alpha_L}$; $\quad \alpha_L \equiv \gamma_n\gamma_{L,n}$	$\forall n,\ t \in [T,\infty)$
Stationary	$\frac{R_{n,t+1}}{P_{n,t+1}} = \frac{R_{n,t}}{P_{n,t}}$; $\quad 1+r_{n,t+1} = 1+r_{n,t}$; $\quad s^R_{n,g,t+1} = s^R_{n,g,t}$; $\quad \pi_{ni,t+1} = \pi_{ni,t}$	$\forall n, i, g,\ t \in [T,\infty)$
Knowledge value	$\frac{v_{n,t+1}}{P_{n,t+1}} = \frac{(1+g_\omega)}{(1+g_\lambda)^\nu}\frac{v_{n,t}}{P_{n,t}}$; $\quad \frac{(1+g_\omega)/(1+g_\lambda)^\nu}{1+r_{n,T}} < 1$	$\forall n,\ t \in [T,\infty)$
Nominal growth	$\sum_{n=1}^{N} \mathrm{GDP}_{n,t} = 1$; $\quad (1+g_P) = (1+g_R) = (1+g_K)^{-1}$	$\forall n,\ t \in [T,\infty)$
	$(1+g_W) = (1+g_{d^F}) = (1+g_{tr^D}) = (1+g_{tr^T}) = (1+g_n)^{-1}$	$\forall n,\ t \in [T,\infty)$
	$(1+g_c) = \frac{1+g_A}{1+g_K}$; $\quad (1+g_v) = (1+g_\lambda)^{-1}$	$\forall n,\ t \in [T,\infty)$
	$(1+g_X) = (1+g_Y) = (1+g_D) = 1$	$\forall n,\ t \in [T,\infty)$
	$(1+g_{\Pi^{\mathrm{op}}}) = (1+g_{D^F}) = 1$; $\quad (1+g_\lambda)(1+g_A)^{-\theta} = 1$	$\forall n,\ t \in [T,\infty)$
	$(1+g_P)(1+g_K) = 1$; $\quad (1+g_W)(1+g_n) = 1$; $\quad (1+g_v)(1+g_\lambda) = 1$	$\forall n,\ t \in [T,\infty)$
F1	$(1+g_\lambda)\lambda_{n,T} = \lambda_{n,T} + \bar{\alpha}_n\lambda^\nu_{n,T}\sum_{g=1}^G \eta_g \left(s^R_{n,g,T}\right)^\xi L_{n,g,T}$	$\forall n$
F1′	$\lambda_{n,T} = \left[\frac{\bar{\alpha}_n}{g_\lambda}\sum_{g=1}^G \eta_g \left(s^R_{n,g,T}\right)^\xi L_{n,g,T}\right]^{\frac{1}{1-\nu}}$	$\forall n:\ (1+g_\lambda) > 1$
F2	$L^R_{n,g,T} = s^R_{n,g,T}L_{n,g,T}$; $\quad L^P_{n,g,T} = (1-s^R_{n,g,T})L_{n,g,T}$; $\quad \mathcal{L}^R_{n,T} = \sum_g l_{n,g}L^R_{n,g,T}$; $\quad \mathcal{L}^P_{n,T} = \sum_g l_{n,g}L^P_{n,g,T}$	$\forall n, g$
F2′	$s^R_{n,g,T} = \min\left\{1, \left(\frac{\xi v_{n,T}\bar{\alpha}_n\lambda^\nu_{n,T}\eta_g}{W_{n,T}l_{n,g}}\right)^{\frac{1}{1-\xi}}\right\}$; $\quad \xi v_{n,T}\bar{\alpha}_n\lambda^\nu_{n,T}\eta_g\left(s^R_{n,g,T}\right)^{\xi-1} - W_{n,T}l_{n,g} \begin{cases} = 0, & s^R_{n,g,T} < 1, \\ \geq 0, & s^R_{n,g,T} = 1 \end{cases}$	$\forall n, g:\ l_{n,g} > 0$
F3	$W_{n,T}\mathcal{L}^P_{n,T} = \gamma_n\gamma_{L,n}\frac{\theta}{1+\theta}Y_{n,T}$; $\quad R_{n,T}K_{n,T} = \gamma_n(1-\gamma_{L,n})\frac{\theta}{1+\theta}Y_{n,T}$; $\quad P_{n,T}M_{n,T} = (1-\gamma_n)\frac{\theta}{1+\theta}Y_{n,T}$	$\forall n$
F4	$r_{n,T} = \frac{R_{n,T}}{P_{n,T}} - \delta$; $\quad \frac{v_{n,T}}{P_{n,T}} = \frac{1}{1+r_{n,T}}\left[\frac{1+g_\omega}{(1+g_\lambda)^\nu}\frac{\Pi^{\mathrm{op}}_{n,T}}{P_{n,T}\lambda_{n,T}} + \frac{1+g_\omega}{(1+g_\lambda)^\nu}\frac{v_{n,T}}{P_{n,T}}\right]$	$\forall n$
F4′	$\frac{v_{n,T}}{P_{n,T}} = \frac{\frac{1+g_\omega}{(1+g_\lambda)^\nu}}{1+r_{n,T} - \frac{1+g_\omega}{(1+g_\lambda)^\nu}}\frac{\Pi^{\mathrm{op}}_{n,T}}{P_{n,T}\lambda_{n,T}}$; $\quad \frac{1+g_\omega}{(1+g_\lambda)^\nu} < 1+r_{n,T}$	$\forall n$
H1	$N_{n,T} \equiv \sum_{g=1}^G N_{n,g,T}$; $\quad n_{n,g,T} \equiv \frac{N_{n,g,T}}{N_{n,T}}$; $\quad L_{n,g,T} = (1-\tau^L_{n,T})N_{n,g,T}$; $\quad L^e_{n,T} \equiv \mathcal{L}^P_{n,T} + \mathcal{L}^R_{n,T} = (1-\tau^L_{n,T})\sum_{g=1}^G l_{n,g}N_{n,g,T}$	$\forall n, g$
H2	$P_{n,T}c_{n,g,T} + P_{n,T}(1+g_\omega)a_{n,g+1,T} = P_{n,T}(1+r_{n,T})a_{n,g,T} + W_{n,T}(1-\tau^L_{n,T})l_{n,g} + d^F_{n,T} + tr^D_{n,T} + tr^T_{n,T}$	$\forall n,\ g \in [1,G]$
H2′	$c_{n,g,T} = (1+r_{n,T})a_{n,g,T} - (1+g_\omega)a_{n,g+1,T} + \frac{W_{n,T}}{P_{n,T}}(1-\tau^L_{n,T})l_{n,g} + \frac{d^F_{n,T} + tr^D_{n,T} + tr^T_{n,T}}{P_{n,T}}$	$\forall n,\ g \in [1,G]$
H3	$a_{n,1,T} = a_{n,G+1,T} = 0$; $\quad c_{n,g,T} > 0$; $\quad \{c_{n,g,T}\}_{g=1}^G$; $\quad \{a_{n,g+1,T}\}_{g=1}^{G-1}$	$\forall n$
H4	$tr^T_{n,T} \equiv \frac{D_{n,T}}{N_{n,T}}$; $\quad tr^D_{n,T} = P_{n,T}(1+r_{n,T})\sum_{g=2}^G\left(\frac{n_{n,g-1,T}}{1+g_n} - n_{n,g,T}\right)a_{n,g,T}$; $\quad d^F_{n,T} \equiv \frac{D^F_{n,T}}{N_{n,T}}$	$\forall n$
H4′	$tr^D_{n,T} = tr^{D,1}_{n,T} + tr^{D,2}_{n,T}$; $\quad tr^{D,1}_{n,T} = P_{n,T}(1-\delta)\sum_{g=2}^G\left(\frac{n_{n,g-1,T}}{1+g_n} - n_{n,g,T}\right)a_{n,g,T}$; $\quad tr^{D,2}_{n,T} = R_{n,T}\sum_{g=2}^G\left(\frac{n_{n,g-1,T}}{1+g_n} - n_{n,g,T}\right)a_{n,g,T}$	$\forall n$
H4″	$P_{n,T}c_{n,g,T} + P_{n,T}i_{n,g,T} = R_{n,T}a_{n,g,T} + W_{n,T}(1-\tau^L_{n,T})l_{n,g} + d^F_{n,T} + tr^{D,2}_{n,T} + tr^T_{n,T}$	$\forall n,\ g \in [1,G]$
H4‴	$P_{n,T}i_{n,g,T} = P_{n,T}(1+g_\omega)a_{n,g+1,T} - \left[P_{n,T}(1-\delta)a_{n,g,T} + tr^{D,1}_{n,T}\right]$	$\forall n,\ g \in [1,G]$
H5	$u'(c_{n,g,T}) = \beta s_{n,g+1,T+1}\psi_n(1+r_{n,T+1})u'\left((1+g_\omega)c_{n,g+1,T}\right)$	$\forall n,\ g \in [1,G-1]$
H5′	$(1+g_\omega)\frac{c_{n,g+1,T}}{c_{n,g,T}} = \left[\beta s_{n,g+1,T+1}\psi_n(1+r_{n,T+1})\right]^\rho$	$\forall n,\ g \in [1,G-1]$
H6	$C_{n,T} \equiv \sum_{g=1}^G N_{n,g,T}c_{n,g,T}$; $\quad K_{n,T} \equiv \sum_{g=2}^G \frac{N_{n,g-1,T}}{(1+g_n)}a_{n,g,T} = N_{n,T}\sum_{g=2}^G\frac{n_{n,g-1,T}}{(1+g_n)}a_{n,g,T}$	$\forall n$
T1	$c_{n,T} = \Upsilon_n\left(W^{\gamma_{L,n}}_{n,T}R^{1-\gamma_{L,n}}_{n,T}\right)^{\gamma_n}P^{1-\gamma_n}_{n,T}$; $\quad \Upsilon_n = (\gamma_n\gamma_{L,n})^{-\gamma_n\gamma_{L,n}}\left[\gamma_n(1-\gamma_{L,n})\right]^{-\gamma_n(1-\gamma_{L,n})}(1-\gamma_n)^{-(1-\gamma_n)}$	$\forall n$
T2	$p_{n,T}(\omega) = C_{2,n,T}(\omega)$; $\quad P_{n,T} = \mathcal{B}(\theta,\sigma)\left[\sum_{i=1}^N \lambda_{i,T}(c_{i,T}\kappa_{ni,T})^{-\theta}\right]^{-1/\theta}$; $\quad \mathcal{B}(\theta,\sigma) = \Gamma\left(2 - \frac{\sigma-1}{\theta}\right)^{\frac{1}{1-\sigma}}$	$\forall n, \omega$
T3	$\pi_{ni,T} \equiv \frac{X_{ni,T}}{X_{n,T}} = \frac{\lambda_{i,T}(c_{i,T}\kappa_{ni,T})^{-\theta}}{\sum_{m=1}^N \lambda_{m,T}(c_{m,T}\kappa_{nm,T})^{-\theta}}$; $\quad X_{ni,T} = \pi_{ni,T}X_{n,T}$	$\forall n, i$
T4	$P_{n,T}C_{n,T} + P_{n,T}I_{n,T} = W_{n,T}\mathcal{L}^P_{n,T} + W_{n,T}\mathcal{L}^R_{n,T} + R_{n,T}K_{n,T} + D^F_{n,T} + D_{n,T}$	$\forall n$
T4′	$P_{n,T}C_{n,T} + P_{n,T}I_{n,T} = W_{n,T}\mathcal{L}^P_{n,T} + R_{n,T}K_{n,T} + \Pi^{\mathrm{op}}_{n,T} + D_{n,T}$; $\quad Y_{n,T} = W_{n,T}\mathcal{L}^P_{n,T} + R_{n,T}K_{n,T} + P_{n,T}M_{n,T} + \Pi^{\mathrm{op}}_{n,T}$	$\forall n$
T5	$(1+g_K)K_{n,T} = I_{n,T} + (1-\delta)K_{n,T}$	$\forall n$
T5′	$I_{n,T} = (g_K+\delta)K_{n,T}$	$\forall n$
T6	$\sum_{i=1}^N X_{in,T} - \sum_{i=1}^N X_{ni,T} = NX_{n,T} = -D_{n,T}$; $\quad Y_{n,T} \equiv \sum_{i=1}^N X_{in,T}$	$\forall n$
T6′	$Q_{n,T} = C_{n,T} + I_{n,T} + M_{n,T}$; $\quad X_{n,T} = P_{n,T}Q_{n,T} = Y_{n,T} + D_{n,T}$; $\quad \Pi^{\mathrm{op}}_{n,T} = \frac{1}{1+\theta}Y_{n,T}$	$\forall n$
T7	$D^F_{n,T} = \Pi^{\mathrm{op}}_{n,T} - W_{n,T}\mathcal{L}^R_{n,T}$; $\quad \mathrm{GDP}_{n,T} \equiv W_{n,T}L^e_{n,T} + R_{n,T}K_{n,T} + D^F_{n,T} = W_{n,T}\mathcal{L}^P_{n,T} + R_{n,T}K_{n,T} + \Pi^{\mathrm{op}}_{n,T}$	$\forall n$
T7′	$\sum_{n=1}^N \mathrm{GDP}_{n,T} = 1$	
T8	$D_{n,T} = -\phi_{n,T}\left(R_{n,T}K_{n,T} + W_{n,T}L^e_{n,T}\right) + N_{n,T}T^P_T$; $\quad T^P_T \equiv \frac{\sum_{m=1}^N \phi_{m,T}\left(R_{m,T}K_{m,T} + W_{m,T}L^e_{m,T}\right)}{\sum_{m=1}^N N_{m,T}}$	$\forall n$
T8′	$D_{n,T} = -\phi_{n,T}\left(R_{n,T}K_{n,T} + W_{n,T}L^e_{n,T}\right) + \frac{N_{n,T}}{\sum_{m=1}^N N_{m,T}}\sum_{m=1}^N \phi_{m,T}\left(R_{m,T}K_{m,T} + W_{m,T}L^e_{m,T}\right)$; $\quad \sum_{n=1}^N D_{n,T} = 0$	$\forall n$

Table 3: Dynamic transitional equilibrium

F1	$\lambda_{n,t+1} - \lambda_{n,t} = \bar{\alpha}_n \lambda_{n,t}^{\nu} \sum_{g=1}^{G} \eta_g \left(s_{n,g,t}^R\right)^{\xi} L_{n,g,t}$	$\forall n,t$
F2	$L_{n,g,t}^R = s_{n,g,t}^R L_{n,g,t}; \quad L_{n,g,t}^P = (1 - s_{n,g,t}^R) L_{n,g,t}; \quad \mathcal{L}_{n,t}^R = \sum_g l_{n,g} L_{n,g,t}^R; \quad \mathcal{L}_{n,t}^P = \sum_g l_{n,g} L_{n,g,t}^P$	$\forall n,g,t$
F2′	$s_{n,g,t}^R = \min \left\{ 1, \left(\frac{\xi v_{n,t} \bar{\alpha}_n \lambda_{n,t}^{\nu} \eta_g}{W_{n,t} l_{n,g}} \right)^{\frac{1}{1-\xi}} \right\}; \quad \xi v_{n,t} \bar{\alpha}_n \lambda_{n,t}^{\nu} \eta_g \left(s_{n,g,t}^R\right)^{\xi-1} - W_{n,t} l_{n,g} \begin{cases} = 0, & s_{n,g,t}^R < 1, \\ \geq 0, & s_{n,g,t}^R = 1 \end{cases}$	$\forall n,g,t:\ l_{n,g} > 0$
F3	$W_{n,t} \mathcal{L}_{n,t}^P = \gamma_n \gamma_{L,n} \frac{\theta}{1+\theta} Y_{n,t}; \quad R_{n,t} K_{n,t} = \gamma_n (1 - \gamma_{L,n}) \frac{\theta}{1+\theta} Y_{n,t}; \quad P_{n,t} M_{n,t} = (1 - \gamma_n) \frac{\theta}{1+\theta} Y_{n,t}$	$\forall n,t$
F4	$r_{n,t} = \frac{R_{n,t}}{P_{n,t}} - \delta; \quad v_{n,t} = \frac{P_{n,t}}{P_{n,t+1}(1 + r_{n,t+1})} \left[\frac{\Pi_{n,t+1}^{\text{op}}}{\lambda_{n,t+1}} + v_{n,t+1} \right]; \quad \lim_{T\to\infty} m_{n,t,T}^K v_{n,T} = 0$	$\forall n,t$
H1	$N_{n,t} \equiv \sum_{g=1}^{G} N_{n,g,t}; \quad n_{n,g,t} \equiv \frac{N_{n,g,t}}{N_{n,t}}; \quad L_{n,g,t} = (1 - \tau_{n,t}^L) N_{n,g,t}; \quad L_{n,t}^e \equiv \mathcal{L}_{n,t}^P + \mathcal{L}_{n,t}^R = (1 - \tau_{n,t}^L) \sum_{g=1}^{G} l_{n,g} N_{n,g,t}$	$\forall n,g,t$
H2	$P_{n,t} c_{n,g,t} + P_{n,t} a_{n,g+1,t+1} = P_{n,t}(1 + r_{n,t}) a_{n,g,t} + W_{n,t}(1 - \tau_{n,t}^L) l_{n,g} + d_{n,t}^F + tr_{n,t}^D + tr_{n,t}^T$	$\forall n,t,\ g \in [1,G]$
H2′	$c_{n,g,t} = (1 + r_{n,t}) a_{n,g,t} - a_{n,g+1,t+1} + \frac{W_{n,t}}{P_{n,t}}(1 - \tau_{n,t}^L) l_{n,g} + \frac{d_{n,t}^F + tr_{n,t}^D + tr_{n,t}^T}{P_{n,t}}$	$\forall n,t,\ g \in [1,G]$
H3	$a_{n,1,b} = 0; \quad a_{n,G+1,b+G} = 0; \quad c_{n,g,b+g-1} > 0; \quad \{c_{n,g,b+g-1}\}_{g=1}^{G}; \quad \{a_{n,g+1,b+g}\}_{g=1}^{G-1}$	$\forall n,b$
H4	$tr_{n,t}^T \equiv \frac{D_{n,t}}{N_{n,t}}; \quad tr_{n,t}^D = P_{n,t}(1 + r_{n,t}) \frac{\sum_{g=2}^{G} \left(N_{n,g-1,t-1} - N_{n,g,t}\right) a_{n,g,t}}{N_{n,t}}; \quad d_{n,t}^F \equiv \frac{D_{n,t}^F}{N_{n,t}}$	$\forall n,t$
H4′	$tr_{n,t}^D = tr_{n,t}^{D,1} + tr_{n,t}^{D,2}; \quad tr_{n,t}^{D,1} = P_{n,t}(1 - \delta) \frac{\sum_{g=2}^{G} \left(N_{n,g-1,t-1} - N_{n,g,t}\right) a_{n,g,t}}{N_{n,t}}; \quad tr_{n,t}^{D,2} = R_{n,t} \frac{\sum_{g=2}^{G} \left(N_{n,g-1,t-1} - N_{n,g,t}\right) a_{n,g,t}}{N_{n,t}}$	$\forall n,t$
H4″	$P_{n,t} c_{n,g,t} + P_{n,t} i_{n,g,t} = R_{n,t} a_{n,g,t} + W_{n,t}(1 - \tau_{n,t}^L) l_{n,g} + d_{n,t}^F + tr_{n,t}^{D,2} + tr_{n,t}^T$	$\forall n,t,\ g \in [1,G]$
H4‴	$P_{n,t} i_{n,g,t} = P_{n,t} a_{n,g+1,t+1} - \left[P_{n,t}(1 - \delta) a_{n,g,t} + tr_{n,t}^{D,1}\right]$	$\forall n,t,\ g \in [1,G]$
H5	$u'(c_{n,g,t}) = \beta s_{n,g+1,t+1} \psi_{n,t+1} (1 + r_{n,t+1}) u'(c_{n,g+1,t+1})$	$\forall n,t,\ g \in [1,G-1]$
H5′	$\frac{c_{n,g+1,t+1}}{c_{n,g,t}} = \left[\beta s_{n,g+1,t+1} \psi_{n,t+1} (1 + r_{n,t+1})\right]^{\rho}$	$\forall n,t,\ g \in [1,G-1]$
H6	$C_{n,t} \equiv \sum_{g=1}^{G} N_{n,g,t} c_{n,g,t}; \quad K_{n,t} \equiv \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}; \quad K_{n,t+1} \equiv \sum_{g=1}^{G-1} N_{n,g,t} a_{n,g+1,t+1}$	$\forall n,t$
T1	$c_{n,t} = \Upsilon_n \left(W_{n,t}^{\gamma_{L,n}} R_{n,t}^{1-\gamma_{L,n}} \right)^{\gamma_n} P_{n,t}^{1-\gamma_n}; \quad \Upsilon_n = (\gamma_n \gamma_{L,n})^{-\gamma_n \gamma_{L,n}} [\gamma_n (1 - \gamma_{L,n})]^{-\gamma_n (1-\gamma_{L,n})} (1 - \gamma_n)^{-(1-\gamma_n)}$	$\forall n,t$
T2	$p_{n,t}(\omega) = C_{2,n,t}(\omega); \quad P_{n,t} = \mathcal{B}(\theta,\sigma) \left[\sum_{i=1}^{N} \lambda_{i,t} (c_{i,t} \kappa_{ni,t})^{-\theta}\right]^{-1/\theta}; \quad \mathcal{B}(\theta,\sigma) = \Gamma\left(2 - \frac{\sigma - 1}{\theta}\right)^{\frac{1}{1-\sigma}}$	$\forall n,t,\omega$
T3	$\pi_{ni,t} \equiv \frac{X_{ni,t}}{X_{n,t}} = \frac{\lambda_{i,t} (c_{i,t} \kappa_{ni,t})^{-\theta}}{\sum_{m=1}^{N} \lambda_{m,t} (c_{m,t} \kappa_{nm,t})^{-\theta}}; \quad X_{ni,t} = \pi_{ni,t} X_{n,t}$	$\forall n,i,t$
T4	$P_{n,t} C_{n,t} + P_{n,t} I_{n,t} = W_{n,t} \mathcal{L}_{n,t}^P + W_{n,t} \mathcal{L}_{n,t}^R + R_{n,t} K_{n,t} + D_{n,t}^F + D_{n,t}$	$\forall n,t$
T4′	$P_{n,t} C_{n,t} + P_{n,t} I_{n,t} = W_{n,t} \mathcal{L}_{n,t}^P + R_{n,t} K_{n,t} + \Pi_{n,t}^{\text{op}} + D_{n,t}; \quad Y_{n,t} = W_{n,t} \mathcal{L}_{n,t}^P + R_{n,t} K_{n,t} + P_{n,t} M_{n,t} + \Pi_{n,t}^{\text{op}}$	$\forall n,t$
T5	$K_{n,t+1} = I_{n,t} + (1 - \delta) K_{n,t}$	$\forall n,t$
T6	$\sum_{i=1}^{N} X_{in,t} - \sum_{i=1}^{N} X_{ni,t} = NX_{n,t} = -D_{n,t}; \quad Y_{n,t} \equiv \sum_{i=1}^{N} X_{in,t}$	$\forall n,t$
T6′	$Q_{n,t} = C_{n,t} + I_{n,t} + M_{n,t}; \quad X_{n,t} = P_{n,t} Q_{n,t} = Y_{n,t} + D_{n,t}; \quad \Pi_{n,t}^{\text{op}} = \frac{1}{1+\theta} Y_{n,t}$	$\forall n,t$
T7	$D_{n,t}^F = \Pi_{n,t}^{\text{op}} - W_{n,t} \mathcal{L}_{n,t}^R; \quad \text{GDP}_{n,t} \equiv W_{n,t} L_{n,t}^e + R_{n,t} K_{n,t} + D_{n,t}^F = W_{n,t} \mathcal{L}_{n,t}^P + R_{n,t} K_{n,t} + \Pi_{n,t}^{\text{op}}$	$\forall n,t$
T7′	$\sum_{n=1}^{N} \text{GDP}_{n,t} = 1$	$\forall t$
T8	$D_{n,t} = -\phi_{n,t} \left(R_{n,t} K_{n,t} + W_{n,t} L_{n,t}^e\right) + N_{n,t} T_t^P; \quad T_t^P \equiv \frac{\sum_{m=1}^{N} \phi_{m,t} \left(R_{m,t} K_{m,t} + W_{m,t} L_{m,t}^e\right)}{\sum_{m=1}^{N} N_{m,t}}$	$\forall n,t$
T8′	$D_{n,t} = -\phi_{n,t} \left(R_{n,t} K_{n,t} + W_{n,t} L_{n,t}^e\right) + \frac{N_{n,t}}{\sum_{m=1}^{N} N_{m,t}} \sum_{m=1}^{N} \phi_{m,t} \left(R_{m,t} K_{m,t} + W_{m,t} L_{m,t}^e\right); \quad \sum_{n=1}^{N} D_{n,t} = 0$	$\forall n,t$

A.2 Computational algorithm

A.2.1 Algorithm to Compute the Steady-state

Algorithm 1 Open Economy Steady State: Single-sector Model

1: Guess a vector of capital stocks $\boldsymbol{K}^i = \{K_1, \ldots, K_N\}^i$. Then calculate $\boldsymbol{I}^i = \{I_1, \ldots, I_N\}^i$ from $I_n = (g_K + \delta)K_n, \forall n$.

2: Solve the single-sector production, trade, innovation, and valuation blocks under fixed world GDP. Using the current manuscript's equilibrium conditions, obtain

$$\left\{\pi_{ni}, P_n, I_n, R_n, W_n, X_n, Y_n, \Pi_n^{\text{op}}, D_n^F, \lambda_n, v_n, s_{n,g}^R\right\}, \qquad \forall n, i, g.$$

3: Solve the $(G-1)$ Euler equations for each country n, together with the age-specific budget constraints and asset boundaries, to obtain

$$\left\{a_{n,1}^i = a_{n,G+1}^i = 0, \left\{a_{n,g+1}^i\right\}_{g=1}^{G-1}\right\}.$$

4: Calculate

$$K_n^{i\prime} = \sum_{g=2}^{G} \frac{N_{n,g-1,T}}{1+g_n} a_{n,g}^i$$

for each country n.

5: Check the error term. If

$$\left\|\boldsymbol{K}^i - \boldsymbol{K}^{i\prime}\right\| < \epsilon,$$

stop. Otherwise, go to the next step.

6: Return to Step 2 with the updated guess

$$\boldsymbol{K}^{i+1} = \zeta \boldsymbol{K}^{i\prime} + (1-\zeta)\boldsymbol{K}^i, \qquad \zeta \in (0,1).$$

Detail for Step 3 :

The stationary Euler equation in the current single-sector model is

$$\frac{(1+g_\omega)c_{n,g+1,T}}{c_{n,g,T}} = \left[\beta \psi_n s_{n,g+1,T+1}\left(1 + \frac{R_{n,T}}{P_{n,T}} - \delta\right)\right]^\rho.$$

Equivalently,

$$\frac{(1+g_\omega)c_{n,g+1}}{c_{n,g}} = \left[\beta S_{n,g+1}\Theta_n\right]^\rho,$$

where

$$\Theta_n \equiv 1 + \frac{R_n}{P_n} - \delta = 1 + r_n, \qquad S_{n,g+1} \equiv \psi_n s_{n,g+1}.$$

Thus,

$$P_n c_{n,g} = (1+g_\omega)\left(\beta S_{n,g+1}\Theta_n\right)^{-\rho} P_n c_{n,g+1}.$$

Define

$$\Lambda_{n,g+1} \equiv (1+g_\omega)\left(\beta S_{n,g+1}\Theta_n\right)^{-\rho}.$$

Then

$$P_n c_{n,g} = \Lambda_{n,g+1} P_n c_{n,g+1}.$$

From

$$P_n c_{n,g} = P_n \Theta_n a_{n,g} - P_n(1+g_\omega)a_{n,g+1} + W_n(1-\tau_n^L)l_{n,g} + d_n^F + tr_n^D + tr_n^T$$

and

$$P_n c_{n,g+1} = P_n \Theta_n a_{n,g+1} - P_n(1+g_\omega)a_{n,g+2} + W_n(1-\tau_n^L)l_{n,g+1} + d_n^F + tr_n^D + tr_n^T,$$

we obtain

$$\begin{aligned} &P_n \Theta_n a_{n,g} - P_n(1+g_\omega)a_{n,g+1} + W_n(1-\tau_n^L)l_{n,g} + d_n^F + tr_n^D + tr_n^T \\ &= \Lambda_{n,g+1}\left[P_n \Theta_n a_{n,g+1} - P_n(1+g_\omega)a_{n,g+2} + W_n(1-\tau_n^L)l_{n,g+1} + d_n^F + tr_n^D + tr_n^T\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} &P_n \Theta_n a_{n,g} - \left[\Lambda_{n,g+1}\Theta_n + (1+g_\omega)\right] P_n a_{n,g+1} + \Lambda_{n,g+1}(1+g_\omega)P_n a_{n,g+2} \\ &= W_n(1-\tau_n^L)\left[\Lambda_{n,g+1}l_{n,g+1} - l_{n,g}\right] + (d_n^F + tr_n^T)\left[\Lambda_{n,g+1} - 1\right] + tr_n^D\left[\Lambda_{n,g+1} - 1\right]. \end{aligned} \tag{D.14}$$

The steady-state accidental-bequest transfer is

$$tr_n^D \equiv \frac{P_n \Theta_n}{N_{n,T}} \sum_{g=2}^{G} \left(\frac{N_{n,g-1,T}}{1+g_n} - N_{n,g,T}\right) a_{n,g}.$$

The trade transfer is

$$tr_n^T \equiv \frac{D_{n,T}}{N_{n,T}},$$

where

$$D_n = -\phi_n\left(R_n K_n + W_n L_n^e\right) + \frac{N_n}{\sum_{m=1}^N N_m} \sum_{m=1}^N \phi_m\left(R_m K_m + W_m L_m^e\right).$$

Define

$$\vec{a}_n = \begin{bmatrix} a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,G-1} \\ a_{n,G} \end{bmatrix}_{(G-1)\times 1}, \qquad \vec{l}_n^- = \begin{bmatrix} l_{n,1} \\ l_{n,2} \\ \vdots \\ l_{n,G-2} \\ l_{n,G-1} \end{bmatrix}_{(G-1)\times 1},$$

$$\vec{l}_n^+ = \begin{bmatrix} l_{n,2} \\ l_{n,3} \\ \vdots \\ l_{n,G-1} \\ l_{n,G} \end{bmatrix}_{(G-1)\times 1}, \qquad \vec{N}_n = \begin{bmatrix} N_{n,1,T} \\ N_{n,2,T} \\ \vdots \\ N_{n,G-1,T} \\ N_{n,G,T} \end{bmatrix}_{G\times 1},$$

$$\vec{\Lambda}_n = \begin{bmatrix} \Lambda_{n,2} \\ \Lambda_{n,3} \\ \vdots \\ \Lambda_{n,G-1} \\ \Lambda_{n,G} \end{bmatrix}_{(G-1)\times 1}.$$

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(G-1)\times(G-1)} = \begin{bmatrix} \text{zeros}(1, G-2) & 0 \\ \text{eye}(G-2) & \text{zeros}(G-2, 1) \end{bmatrix},$$

$$A' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(G-1)\times(G-1)}, \qquad I = \text{eye}(G-1).$$

Define the mortality-gap row vector and the corresponding rank-one matrix:

$$\vec{d}_n' \equiv \frac{1}{N_{n,T}} \left(\frac{N_{n,1,T}}{1+g_n} - N_{n,2,T}, \ldots, \frac{N_{n,G-1,T}}{1+g_n} - N_{n,G,T} \right),$$

$$T_n \equiv \text{ones}(G-1, 1)\vec{d}_n'.$$

Then

$$tr_n^D \operatorname{ones}(G-1,1) = P_n \Theta_n T_n \vec{a}_n.$$

Equation (D.14) can be written in matrix form as

$$\vec{a}_n = \text{LeftMatrix}^{-1} \cdot \text{RightVector},$$

where

$$\begin{aligned}\text{LeftMatrix} = {} & P_n \Theta_n A - P_n \operatorname{diag}\left((1+g_\omega) \operatorname{ones}(G-1,1) + \Theta_n \vec{\Lambda}_n\right) \\ & + P_n (1+g_\omega) \operatorname{diag}(\vec{\Lambda}_n) A' - P_n \Theta_n \operatorname{diag}\left(\vec{\Lambda}_n - \operatorname{ones}(G-1,1)\right) T_n,\end{aligned}$$

$$\text{RightVector} = W_n (1-\tau_n^L) \left[\operatorname{diag}(\vec{\Lambda}_n) \vec{l}_n^{+} - \vec{l}_n^{-}\right] + \operatorname{diag}\left(\vec{\Lambda}_n - \operatorname{ones}(G-1,1)\right) (d_n^F + tr_n^T) \operatorname{ones}(G-1,1).$$

Equivalently, the numerically preferred implementation solves

$$\text{LeftMatrix}\, \vec{a}_n = \text{RightVector}$$

directly, rather than explicitly constructing the inverse.

A.2.2 Algorithm to compute the transition path

Algorithm 2 Open Economy Dynamic Equilibrium: Single-sector Model

1: Guess $\overline{\boldsymbol{K}}^1_{t=2,\ldots,T} = \{K_1, \ldots, K_N\}^1_{t=2,\ldots,T}$. The economy reaches a new steady state at $T+1$. The population dynamics reach their steady state before $T+1$.

2: Obtain $\overline{\boldsymbol{I}}^1_{t=1,\ldots,T} = \{I_1, \ldots, I_N\}^1_{t=1,\ldots,T}$, where

$$I^i_{n,t} = K^i_{n,t+1} - (1-\delta)K^i_{n,t}, \qquad \forall n.$$

3: Solve the single-sector production, trade, innovation, and valuation blocks under fixed world GDP, obtaining

$$\left\{\pi_{ni,t}, P_{n,t}, I_{n,t}, R_{n,t}, C_{n,t}, W_{n,t}, X_{n,t}, Y_{n,t}, \Pi^{\text{op}}_{n,t}, D^F_{n,t}, \lambda_{n,t}, v_{n,t}, s^R_{n,g,t}\right\}.$$

4: Guess $\overrightarrow{tr}^{D,i,j}$. Repeat the following until $\overrightarrow{tr}^{D,i,j}$ converges at some j_0, and set $\overrightarrow{tr}^{D,i}_{j_0} = \overrightarrow{tr}^{D,i,j_0}$:

(a) For every $G_0 \in \{1, \ldots, G-1\}$, country n, beginning age $g_0 \in \{1, \ldots, G-1\}$, and beginning year $t_0 \in \{1, \ldots, T\}$, solve the corresponding Euler and budget system:

$$\left\{a_{n,g_0,t_0}, a_{n,g_0+G_0+1,t_0+G_0+1}, \{c_{n,g_0+h,t_0+h}\}^{G_0}_{h=0}, \{a_{n,g_0+h+1,t_0+h+1}\}^{G_0-1}_{h=0}\right\}.$$

(b) Check

$$\left\|\overrightarrow{tr}^{D,i,j} - \overrightarrow{tr}^{D,i,j\prime}\right\| < \varepsilon \times 10^{-2}.$$

If it holds, return $\overrightarrow{tr}^{D,i,j}$ and proceed to Step 5. Otherwise update

$$\overrightarrow{tr}^{D,i,j+1} = \zeta' \, \overrightarrow{tr}^{D,i,j\prime} + (1-\zeta') \, \overrightarrow{tr}^{D,i,j}, \qquad \zeta' \in (0,1),$$

and repeat Step 4(a) with $j = j+1$.

5: For iteration i, calculate the implied capital path $\overline{\boldsymbol{K}}^{i\prime}$ for every country and check

$$\left\|\overline{\boldsymbol{K}}^i - \overline{\boldsymbol{K}}^{i\prime}\right\| < \varepsilon.$$

6: **if** the condition holds **then**

7: Return $\overline{\boldsymbol{K}}^i$ and stop.

8: **else**

9: Update

$$\overline{\boldsymbol{K}}^{i+1} = \zeta\overline{\boldsymbol{K}}^{i\prime} + (1-\zeta)\overline{\boldsymbol{K}}^i, \qquad \zeta \in (0,1),$$

and repeat from Step 2 with $i = i+1$.

10: **end if**

Detail for Step 4 :

In the current single-sector model,

$$\frac{c_{n,g+1,t+1}}{c_{n,g,t}} = \left[\beta\psi_{n,t+1}s_{n,g+1,t+1}\left(1+\frac{R_{n,t+1}}{P_{n,t+1}}-\delta\right)\right]^{\rho}.$$

Define

$$S_{n,g+1,t+1} \equiv \psi_{n,t+1}s_{n,g+1,t+1}, \qquad \Theta_{n,t+1} \equiv 1+\frac{R_{n,t+1}}{P_{n,t+1}}-\delta = 1+r_{n,t+1}.$$

Then

$$\frac{c_{n,g+1,t+1}}{c_{n,g,t}} = \left[\beta S_{n,g+1,t+1}\Theta_{n,t+1}\right]^{\rho}.$$

Consequently,

$$\frac{P_{n,t+1}c_{n,g+1,t+1}}{P_{n,t}c_{n,g,t}} = \frac{P_{n,t+1}}{P_{n,t}}\left[\beta S_{n,g+1,t+1}\Theta_{n,t+1}\right]^{\rho}.$$

Define

$$\Lambda_{n,g+1,t+1} \equiv \frac{P_{n,t}}{P_{n,t+1}}\left[\beta S_{n,g+1,t+1}\Theta_{n,t+1}\right]^{-\rho}.$$

Thus,

$$P_{n,t}c_{n,g,t} = \Lambda_{n,g+1,t+1}P_{n,t+1}c_{n,g+1,t+1}.$$

Substituting the single-sector household budgets gives

$$P_{n,t}\Theta_{n,t}a_{n,g,t} - P_{n,t}a_{n,g+1,t+1} + W_{n,t}(1-\tau^{L}_{n,t})l_{n,g} + d^{F}_{n,t} + tr^{D}_{n,t} + tr^{T}_{n,t}$$

$$= \Lambda_{n,g+1,t+1}\left[P_{n,t+1}\Theta_{n,t+1}a_{n,g+1,t+1} - P_{n,t+1}a_{n,g+2,t+2} + W_{n,t+1}(1-\tau^{L}_{n,t+1})l_{n,g+1} + d^{F}_{n,t+1} + tr^{D}_{n,t+1} + tr^{T}_{n,t+1}\right].$$

If these equations are solved from (g,t) through $(g+G_0-1, t+G_0-1)$, there are G_0 unknown interior assets under G_0 equations, conditional on the boundary values $a_{n,g,t}$ and $a_{n,g+G_0+1,t+G_0+1}$. The first equation is

$$P_{n,t}\Theta_{n,t}a_{n,g,t} - \left[P_{n,t} + \Lambda_{n,g+1,t+1}P_{n,t+1}\Theta_{n,t+1}\right]a_{n,g+1,t+1} + \Lambda_{n,g+1,t+1}P_{n,t+1}a_{n,g+2,t+2}$$

$$= \Lambda_{n,g+1,t+1}W_{n,t+1}(1-\tau^{L}_{n,t+1})l_{n,g+1} - W_{n,t}(1-\tau^{L}_{n,t})l_{n,g}$$

$$+ \left[\Lambda_{n,g+1,t+1}d^{F}_{n,t+1} - d^{F}_{n,t}\right] + \left[\Lambda_{n,g+1,t+1}tr^{D}_{n,t+1} - tr^{D}_{n,t}\right] + \left[\Lambda_{n,g+1,t+1}tr^{T}_{n,t+1} - tr^{T}_{n,t}\right].$$

The last equation is

$$P_{n,t+G_0-1}\Theta_{n,t+G_0-1}a_{n,g+G_0-1,t+G_0-1}$$

$$-\left[P_{n,t+G_0-1} + \Lambda_{n,g+G_0,t+G_0}P_{n,t+G_0}\Theta_{n,t+G_0}\right]a_{n,g+G_0,t+G_0}$$

$$
\begin{aligned}
&+\Lambda_{n,g+G_0,t+G_0} P_{n,t+G_0} a_{n,g+G_0+1,t+G_0+1} \\
&= \Lambda_{n,g+G_0,t+G_0} W_{n,t+G_0}(1-\tau^L_{n,t+G_0}) l_{n,g+G_0} \\
&-W_{n,t+G_0-1}(1-\tau^L_{n,t+G_0-1}) l_{n,g+G_0-1} \\
&+\left[\Lambda_{n,g+G_0,t+G_0} d^F_{n,t+G_0} - d^F_{n,t+G_0-1}\right] \\
&+\left[\Lambda_{n,g+G_0,t+G_0} tr^D_{n,t+G_0} - tr^D_{n,t+G_0-1}\right] + \left[\Lambda_{n,g+G_0,t+G_0} tr^T_{n,t+G_0} - tr^T_{n,t+G_0-1}\right].
\end{aligned}
$$

Define

$$
\overrightarrow{a}_n \equiv \begin{bmatrix} a_{n,g+1,t+1} \\ a_{n,g+2,t+2} \\ \vdots \\ a_{n,g+G_0-1,t+G_0-1} \\ a_{n,g+G_0,t+G_0} \end{bmatrix}_{G_0\times 1},
$$

$$
\overrightarrow{a}^{*}_{n,g,t} \equiv \begin{bmatrix} a_{n,g,t} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{G_0\times 1}, \quad \overrightarrow{a}^{**}_{n,g+G_0+1,t+G_0+1} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_{n,g+G_0+1,t+G_0+1} \end{bmatrix}_{G_0\times 1}.
$$

Let

$$
A \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{G_0\times G_0} = \begin{bmatrix} \text{zeros}(1, G_0-1) & 0 \\ \text{eye}(G_0-1) & \text{zeros}(G_0-1,1) \end{bmatrix},
$$

$$
A' \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{G_0\times G_0}, \quad E \equiv \text{eye}(G_0).
$$

Define

$$\overrightarrow{\Theta P}_n \equiv \begin{bmatrix} \Theta_{n,t} P_{n,t} \\ \Theta_{n,t+1} P_{n,t+1} \\ \vdots \\ \Theta_{n,t+G_0-2} P_{n,t+G_0-2} \\ \Theta_{n,t+G_0-1} P_{n,t+G_0-1} \end{bmatrix}_{G_0 \times 1},$$

$$\overrightarrow{\Lambda P}_n \equiv \begin{bmatrix} \Lambda_{n,g+1,t+1} P_{n,t+1} \\ \Lambda_{n,g+2,t+2} P_{n,t+2} \\ \vdots \\ \Lambda_{n,g+G_0-1,t+G_0-1} P_{n,t+G_0-1} \\ \Lambda_{n,g+G_0,t+G_0} P_{n,t+G_0} \end{bmatrix}_{G_0 \times 1},$$

$$\overrightarrow{P + \Lambda \Theta P}_n \equiv \begin{bmatrix} P_{n,t} + \Lambda_{n,g+1,t+1} \Theta_{n,t+1} P_{n,t+1} \\ P_{n,t+1} + \Lambda_{n,g+2,t+2} \Theta_{n,t+2} P_{n,t+2} \\ \vdots \\ P_{n,t+G_0-2} + \Lambda_{n,g+G_0-1,t+G_0-1} \Theta_{n,t+G_0-1} P_{n,t+G_0-1} \\ P_{n,t+G_0-1} + \Lambda_{n,g+G_0,t+G_0} \Theta_{n,t+G_0} P_{n,t+G_0} \end{bmatrix}_{G_0 \times 1}.$$

The shift matrices imply

$$A \overrightarrow{a}_n + E \overrightarrow{a}^{*}_{n,g,t} = \begin{bmatrix} a_{n,g,t} \\ a_{n,g+1,t+1} \\ \vdots \\ a_{n,g+G_0-2,t+G_0-2} \\ a_{n,g+G_0-1,t+G_0-1} \end{bmatrix}_{G_0 \times 1},$$

$$A' \overrightarrow{a}_n + E \overrightarrow{a}^{**}_{n,g+G_0+1,t+G_0+1} = \begin{bmatrix} a_{n,g+2,t+2} \\ a_{n,g+3,t+3} \\ \vdots \\ a_{n,g+G_0,t+G_0} \\ a_{n,g+G_0+1,t+G_0+1} \end{bmatrix}_{G_0 \times 1}.$$

Define the next- and current-period labor-income vectors

$$\Lambda\overrightarrow{W}\, l_n^+ \equiv \begin{bmatrix} \Lambda_{n,g+1,t+1}W_{n,t+1}(1-\tau^L_{n,t+1})l_{n,g+1} \\ \Lambda_{n,g+2,t+2}W_{n,t+2}(1-\tau^L_{n,t+2})l_{n,g+2} \\ \vdots \\ \Lambda_{n,g+G_0,t+G_0}W_{n,t+G_0}(1-\tau^L_{n,t+G_0})l_{n,g+G_0} \end{bmatrix}_{G_0\times 1},$$

$$\overrightarrow{W}\, l_n^- \equiv \begin{bmatrix} W_{n,t}(1-\tau^L_{n,t})l_{n,g} \\ W_{n,t+1}(1-\tau^L_{n,t+1})l_{n,g+1} \\ \vdots \\ W_{n,t+G_0-1}(1-\tau^L_{n,t+G_0-1})l_{n,g+G_0-1} \end{bmatrix}_{G_0\times 1}.$$

For any age-invariant per-capita payment $z_{n,t} \in \{d^F_{n,t}, tr^D_{n,t}, tr^T_{n,t}\}$, define

$$\Lambda\Delta\overrightarrow{z}_n \equiv \begin{bmatrix} \Lambda_{n,g+1,t+1}z_{n,t+1} - z_{n,t} \\ \Lambda_{n,g+2,t+2}z_{n,t+2} - z_{n,t+1} \\ \vdots \\ \Lambda_{n,g+G_0-1,t+G_0-1}z_{n,t+G_0-1} - z_{n,t+G_0-2} \\ \Lambda_{n,g+G_0,t+G_0}z_{n,t+G_0} - z_{n,t+G_0-1} \end{bmatrix}_{G_0\times 1}.$$

Here,

$$tr^T_{n,t} \equiv \frac{D_{n,t}}{N_{n,t}},$$

where

$$D_{n,t} = -\phi_{n,t}\left(R_{n,t}K_{n,t} + W_{n,t}L^e_{n,t}\right) + \frac{N_{n,t}}{\sum_{m=1}^N N_{m,t}} \sum_{m=1}^N \phi_{m,t}\left(R_{m,t}K_{m,t} + W_{m,t}L^e_{m,t}\right).$$

Also,

$$tr^D_{n,t} \equiv P_{n,t}\Theta_{n,t}\frac{\sum_{g=2}^{G}\left(N_{n,g-1,t-1} - N_{n,g,t}\right)a_{n,g,t}}{N_{n,t}}.$$

In vector form,

$$tr^D_{n,t} = \frac{P_{n,t}\Theta_{n,t}}{N_{n,t}}\left[\left(\overrightarrow{N}_{n,t-1}(1:G-1,1) - \overrightarrow{N}_{n,t}(2:G,1)\right)'\overrightarrow{a}_{n,t}(2:G,1)\right].$$

For a vector x and a conformable matrix B, define

$$x \circ B \equiv \mathrm{diag}(x)B.$$

For two vectors, $\circ$ denotes element-by-element multiplication. Then

$$\text{Leftside} \equiv \overrightarrow{\Theta P}_n \circ \left[A \overrightarrow{a}_n + E \overrightarrow{a}^{*}_{n,g,t} \right] - \overrightarrow{P + \Lambda\Theta P}_n \circ \overrightarrow{a}_n$$

$$+ \overrightarrow{\Lambda P}_n \circ \left[A' \overrightarrow{a}_n + E \overrightarrow{a}^{**}_{n,g+G_0+1,t+G_0+1} \right].$$

Equivalently,

$$\text{Leftside} = \left[\overrightarrow{\Theta P}_n \circ A - \overrightarrow{P + \Lambda\Theta P}_n \circ E + \overrightarrow{\Lambda P}_n \circ A' \right] \overrightarrow{a}_n + \text{Res},$$

where

$$\text{Matrix} \equiv \overrightarrow{\Theta P}_n \circ A - \overrightarrow{P + \Lambda\Theta P}_n \circ E + \overrightarrow{\Lambda P}_n \circ A',$$

$$\text{Res} \equiv \overrightarrow{\Theta P}_n \circ E \overrightarrow{a}^{*}_{n,g,t} + \overrightarrow{\Lambda P}_n \circ E \overrightarrow{a}^{**}_{n,g+G_0+1,t+G_0+1}.$$

The right-hand side is

$$\text{Rightside} \equiv \Lambda \overrightarrow{W} l^{+}_n - \overrightarrow{W} l^{-}_n + \Lambda\Delta \overrightarrow{d}^{F}_n + \Lambda\Delta \overrightarrow{tr}^{D}_n + \Lambda\Delta \overrightarrow{tr}^{T}_n.$$

Therefore,

$$\overrightarrow{a}_n = \text{Matrix}^{-1} \left(\text{Rightside} - \text{Res} \right).$$

In numerical implementation, solve

$$\text{Matrix} \overrightarrow{a}_n = \text{Rightside} - \text{Res}$$

directly instead of explicitly forming the inverse.

One can solve for $\overrightarrow{c}_n$ from

$$\begin{bmatrix} P_{n,t} \\ P_{n,t+1} \\ \vdots \\ P_{n,t+G_0-1} \\ P_{n,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} \circ \begin{bmatrix} c_{n,g,t} \\ c_{n,g+1,t+1} \\ \vdots \\ c_{n,g+G_0-1,t+G_0-1} \\ c_{n,g+G_0,t+G_0} \end{bmatrix}_{(G_0+1)\times 1}$$

$$= \begin{bmatrix} \Theta_{n,t} P_{n,t} a_{n,g,t} \\ \Theta_{n,t+1} P_{n,t+1} a_{n,g+1,t+1} \\ \vdots \\ \Theta_{n,t+G_0-1} P_{n,t+G_0-1} a_{n,g+G_0-1,t+G_0-1} \\ \Theta_{n,t+G_0} P_{n,t+G_0} a_{n,g+G_0,t+G_0} \end{bmatrix}_{(G_0+1)\times 1}$$

$$
- \begin{bmatrix} P_{n,t} a_{n,g+1,t+1} \\ P_{n,t+1} a_{n,g+2,t+2} \\ \vdots \\ P_{n,t+G_0-1} a_{n,g+G_0,t+G_0} \\ P_{n,t+G_0} a_{n,g+G_0+1,t+G_0+1} \end{bmatrix}_{(G_0+1)\times 1}
$$

$$
+ \begin{bmatrix} W_{n,t}(1-\tau^L_{n,t}) l_{n,g} \\ W_{n,t+1}(1-\tau^L_{n,t+1}) l_{n,g+1} \\ \vdots \\ W_{n,t+G_0-1}(1-\tau^L_{n,t+G_0-1}) l_{n,g+G_0-1} \\ W_{n,t+G_0}(1-\tau^L_{n,t+G_0}) l_{n,g+G_0} \end{bmatrix}_{(G_0+1)\times 1}
$$

$$
+ \begin{bmatrix} d^F_{n,t} + tr^D_{n,t} + tr^T_{n,t} \\ d^F_{n,t+1} + tr^D_{n,t+1} + tr^T_{n,t+1} \\ \vdots \\ d^F_{n,t+G_0-1} + tr^D_{n,t+G_0-1} + tr^T_{n,t+G_0-1} \\ d^F_{n,t+G_0} + tr^D_{n,t+G_0} + tr^T_{n,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} .
$$

Finally, retaining the original individual-investment accounting,

$$
\begin{bmatrix} i_{n,g,t} \\ i_{n,g+1,t+1} \\ \vdots \\ i_{n,g+G_0-1,t+G_0-1} \\ i_{n,g+G_0,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} = \begin{bmatrix} a_{n,g+1,t+1} \\ a_{n,g+2,t+2} \\ \vdots \\ a_{n,g+G_0,t+G_0} \\ a_{n,g+G_0+1,t+G_0+1} \end{bmatrix}_{(G_0+1)\times 1}
$$

$$
-(1-\delta) \begin{bmatrix} a_{n,g,t} \\ a_{n,g+1,t+1} \\ \vdots \\ a_{n,g+G_0-1,t+G_0-1} \\ a_{n,g+G_0,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} - \begin{bmatrix} tr^{D,1}_{n,t} \\ tr^{D,1}_{n,t+1} \\ \vdots \\ tr^{D,1}_{n,t+G_0-1} \\ tr^{D,1}_{n,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} \oslash \begin{bmatrix} P_{n,t} \\ P_{n,t+1} \\ \vdots \\ P_{n,t+G_0-1} \\ P_{n,t+G_0} \end{bmatrix}_{(G_0+1)\times 1} .
$$

Here $\oslash$ denotes element-by-element division. If the implementation does not separately remove age-cell bequest accounting from individual investment, set $tr^{D,1}_{n,t} = 0$.

A.3 Calibration of saving wedges

A.3.1 Steady-state saving wedge

The steady-state computation jointly solves the demographic, household, production, trade, innovation, valuation, and market-clearing blocks. The saving wedge is calibrated so that the model reproduces the observed steady-state capital stock.

1. Guess the steady-state saving wedges

$$\boldsymbol{\psi}^{(j)} = \left\{\psi_1^{(j)}, \ldots, \psi_N^{(j)}\right\}.$$

2. Given the steady-state demographic structure, prices, wages, returns, transfers, capital stocks, knowledge stocks, and the trial saving wedges, solve the $G-1$ Euler equations, the age-specific budget constraints, and the asset boundary conditions

$$a_{n,1} = 0, \qquad a_{n,G+1} = 0$$

for the stationary age profiles
$$\left\{c_{n,g}^{(j)}, a_{n,g+1}^{(j)}\right\}_{g=1}^{G-1}.$$

3. Compute the implied steady-state capital stock:

$$\widetilde{K}_n^{(j)} = \sum_{g=2}^{G} \frac{N_{n,g-1}}{1+g_n} a_{n,g}^{(j)}. \tag{188}$$

4. If
$$\left\|\widetilde{\mathbf{K}}^{(j)} - \mathbf{K}^{\text{data}}\right\| < \varepsilon_K,$$

stop. Otherwise, update the saving wedges according to

$$\psi_n^{(j+1)} = \psi_n^{(j)} + \zeta_\psi \frac{\widetilde{K}_n^{(j)} - K_n^{\text{data}}}{K_n^{\text{data}}}, \qquad \zeta_\psi \in (0,1), \tag{189}$$

and return to Step 2.

5. Conditional on the calibrated saving wedges, solve the remaining steady-state conditions for capital, research-assignment shares, knowledge, prices, trade shares, factor returns, corporate payouts, transfers, and the value of knowledge until all equilibrium residuals are below tolerance.

A.3.2 Transition-path saving wedges

The transition saving wedges $\{\psi_{n,t}\}$ are calibrated to match the observed path of aggregate capital. The initial wedge $\psi_{n,1}$ is fixed at its calibrated steady-state value.

1. Guess a path
$$\psi_t^{(j)} = \left\{\psi_{1,t}^{(j)}, \ldots, \psi_{N,t}^{(j)}\right\}, \qquad t = 2, \ldots, T+1.$$

2. Given the paths of prices, wages, capital returns, payouts, transfers, survival probabilities, and the trial saving wedges, solve the age-specific budget constraints and Euler equations for every country, birth year, and age. The solution includes
$$\left\{c_{n,g,t}^{(j)}, a_{n,g+1,t+1}^{(j)}\right\}$$
together with the newborn and terminal asset conditions.

3. Aggregate the implied household assets:
$$\widetilde{K}_{n,t}^{(j)} = \sum_{g=2}^{G} N_{n,g-1,t-1} a_{n,g,t}^{(j)}, \qquad t = 2, \ldots, T+1. \tag{190}$$

4. If
$$\left\|\widetilde{\mathbf{K}}_{2:T+1}^{(j)} - \mathbf{K}_{2:T+1}^{\text{data}}\right\| < \varepsilon_K,$$
stop. Otherwise, update
$$\psi_{n,t}^{(j+1)} = \psi_{n,t}^{(j)} + \zeta_\psi \frac{\widetilde{K}_{n,t}^{(j)} - K_{n,t}^{\text{data}}}{K_{n,t}^{\text{data}}}, \qquad \zeta_\psi \in (0,1), \quad t = 2, \ldots, T+1, \tag{191}$$
and return to Step 2.