Notes on CES

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February 13, 2025

Consider a general CES function:

$$Y = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j^{\frac{1}{\eta}} Y_j^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}, \tag{1}$$

where $A, \eta > 0$, $\alpha_j, Y_j > 0 \; \forall \; j$, and $\sum_{j=1}^{\mathcal{J}} \alpha_j = 1$.

1 Case

$$Y = A\left(\sum_{j=1}^{J} \alpha_j Y_j\right)^{\frac{1}{\gamma}}$$

The Linear Case

$$\lim_{\gamma\to 1} Y = A\sum_{j=1}^{\mathcal{J}} Y_j.$$

The Cobb-Douglas Case

$$\lim_{\gamma\to 0} Y = A\frac{1}{\prod_{j=1}^{\mathcal{J}} \alpha_j^{\alpha_j}} \prod_{j=1}^{\mathcal{J}} Y_j^{\alpha_j}.$$

The Leontief Case

$$\lim_{\gamma\to -\infty} Y = A\min\left\{\frac{Y_1}{\alpha_1}, \ldots, \frac{Y_{\mathcal{J}}}{\alpha_{\mathcal{J}}}\right\}.$$

2 Some Law

$$\sum_{j=1}^{\mathcal{J}} P_j Y_j \leq Z, \tag{2}$$

where Z is total money spent. Let us state the Lagrangian (where Λ is the multiplier for the constraint):

Also

$$\sum_{j=1}^{\mathcal{J}} P_j Y_j \equiv PY$$

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$$\mathcal{L} = A\left(\sum_{j=1}^{\mathcal{J}} \alpha_j^{\frac{1}{\eta}} Y_j^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} - \Lambda\left(\sum_{j=1}^{\mathcal{J}} P_j Y_j - Z\right) \tag{3}$$

The necessary first-order condition is

$$A^{\frac{\eta-1}{\eta}} Y^{\frac{1}{\eta}} \alpha_j^{\frac{1}{\eta}} Y_j^{\frac{\eta-1}{\eta}} = \Lambda P_j. \tag{4}$$

This must hold for all $j \in \mathcal{J}$, so we can write

$$Y_j = \frac{\alpha_j}{\alpha_i}\left(\frac{P_j}{P_i}\right)^{-\eta} Y_i \quad \forall\ i, j \tag{5}$$

Also, one can get

$$P = \frac{1}{A}\left(\sum_{j=1}^{\mathcal{J}} \alpha_j P_j^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{6}$$

$$\frac{Y_i}{Y} = \alpha_i \left(\frac{P_i}{P}\right)^{-\eta} A^{\eta-1}. \tag{7}$$

Define the elasticity of substitution as the percentage rise in $\frac{Y_j}{Y_i}$ following a one percent rise in the relative price $\frac{P_i}{P_j}$. It is clear from (3) that

$$\frac{Y_j}{Y_i} = \frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta},$$

so we can write

$$\frac{\partial\left(\frac{Y_j}{Y_i}\right)}{\partial\left(\frac{P_i}{P_j}\right)} = \eta\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta-1}.$$

By dividing by $\left(\frac{Y_j}{Y_i}\right)/\left(\frac{P_i}{P_j}\right)$, a constant **elasticity of substitution** emerges:

$$\text{Elasticity of substitution} \equiv \frac{\frac{\partial\left(\frac{Y_j}{Y_i}\right)}{\partial\left(\frac{P_i}{P_j}\right)}}{\frac{Y_j}{Y_i}/\frac{P_i}{P_j}} = \frac{\eta\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta-1}}{\frac{\alpha_j}{\alpha_i}\left(\frac{P_i}{P_j}\right)^{\eta}} = \eta$$

Defined Z as the income spent on Y. The income elasticity follows from (5):

$$\text{Elasticity of income} \equiv \frac{\frac{\partial Y_j}{Y_j}}{\frac{\partial Z}{Z}} = \frac{\alpha_i\left(\frac{P_i}{P}\right)^{-\eta}\frac{1}{A^{1-\eta}P}}{\alpha_i\left(\frac{P_i}{P}\right)^{-\eta}\frac{1}{A^{1-\eta}P}} = 1$$

3 Utility

Consider the isoelastic utility function

$$U = \frac{C^{1-\sigma}-1}{1-\sigma},$$

with $C > 0$.

$$\lim_{\sigma \to 1} U = \ln(C).$$