Stochastic Process

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1 Stochastic process

1.1 Poisson Process

1. Random Process or Stochastic Process A random process is a collection of random variables usually indexed by time (or sometimes by space).

A **Continuous-time Random Process** is a random process $X(t), t \in \mathcal{J}$ where $\mathcal{J}$ is an interval on the real line.

A **Discrete-time Random Process** (or a random sequence) is a random process $X(n), n \in \mathcal{J}$, where $\mathcal{J}$ is a countable set. Since J is countable, we can write $\mathcal{J} = t_1, t_2, \cdots$.

If the random variable $X(t)$ is a discrete random variable. Then, $X(t)$ is a Discrete-valued Random Process. Discrete-time processes are sometimes obtained from continuous-time processes by discretizing time (sampling at specific times).

2. Counting Process A random process $\{N(t), t \in [0, +\infty)\}$ is said to be a counting process if $N(t)$ is the number of events that occurred from time 0 up to and including time t. For a counting process, we assume

1. $N(0) = 0$;
2. $N(t) \in \{0, 1, 2, \cdots\}$, for all $t \in [0, +\infty)$;
3. for $0 \leq s < t$, $N(t) - N(s)$ shows the number of events that occur in the interval $(s, t]$.

We usually refer to the occurrence of each event as an "**arrival**". For example, if $N(t)$ is the number of accidents in a city up to time t, we still refer to each accident as an arrival.

3. Independent Increments A random process $\{N(t), t \in [0, +\infty)\}$ be a continuous-time random process. We say that $N(t)$ has independent increments if, for all $0 \leq t_1 \cdots \leq t_n$, the random variables

$$N(t_2) - N(t_1), \cdots, N(t_n) - N(t_{n-1})$$

are independent.

4. Stationary Increments Let $\{X(t), t \in [0, +\infty)\}$ be a continuous-time random process. We say that $X(t)$ has stationary increments if, for all $t_2 > t_1 \geq 0$, and all $r > 0$, the two random variables $X(t_2) - X(t_1)$ and $X(t_2 + r) - X(t_1 + r)$ have the same distributions. In other words, the distribution of the difference depends only on the length of the interval $(t1, t2]$ and not on the exact location of the interval on the real line.

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4. Poisson A discrete random variable X is said to be a **Poisson random variable** with parameter μ, shown as $X \sim Poisson(\mu)$, if its range is $R_X = 0, 1, 2, 3, ...$, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\mu}\mu^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

Key facts:

1. if $X \sim Poisson(\mu)$, the $EX = DX = \mu$;
2. if $X_i \sim Poisson(\mu_i)$, the $\sum X_i \sim Poisson(\sum \mu_i)$;
3. the limit of a binomial distribution is the Poisson distribution.

The Poisson process is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate , but completely at random (without a certain structure). For example, suppose that from historical data, we know that earthquakes occur in a certain area at a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random.

5. The limit of a binomial distribution Let $Y_n \sim Binomial(n, p = p(n))$. Let $\mu > 0$ be a fixed real number and $lim_{n\to\infty} np = \mu$. Then, the PMF of Y_n converges to a $Poisson(\mu)$ PMF, as $n \to \infty$. That is, for any $k \in \{0, 1, 2, ...\}$, we have

$$\lim_{n\to\infty} P_{Y_n}(k) = \frac{e^{-\mu}\mu^k}{k!}.$$

$$\mathrm{P}(N(t) = k) = lim_{n\to\infty} \mathcal{C}_n^k (\frac{\lambda t}{n})^k (1 - \frac{\lambda t}{n})^{n-k}.$$

, where $p \equiv \frac{t\lambda}{n}$.

6. The Poisson Process Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, +\infty)\}$ is called a Poisson process with rate λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. the number of arrivals in any interval of length $\tau > 0$ has $Poisson(\lambda\tau)$ distribution.

Notes: The distribution of the number of arrivals at any interval depends only on the length of the interval and not on the exact location of the interval on the real line. Therefore, the Poisson process has stationary increments.

7. Interarrival Times for Poisson Processes If $N(t)$ is a Poisson process with rate λ, then the interarrival times $X_1, X_2, \cdots$ are independent and $X_i \sim Exponential(\lambda)$, for $i = 1, 2, 3, \cdots$.

Let $N(t)$ be a Poisson process with rate λ. Let X_1 be the time of the first arrival. Then,

$$P(X_1 > t) = P(\text{no arrival in } (0, t]) = e^{-\lambda t}.$$

$$F_{X_1}(t) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $X_1 \sim Exponential(\lambda)$. Let X_2 be the time elapsed between the first and the second arrival, then $X_2 \sim Exponential(\lambda)$.

Thinking of the Poisson process, the memoryless property of the interarrival times is consistent with the independent increment property of the Poisson distribution. In some sense, both imply that the number of arrivals in non-overlapping intervals is independent.

7A. Key facts of Exponential distribution If i.i.d $X_i \sim Exponential(\lambda_i)$:

1. $E(X_i) = 1/\lambda_i$, $D(X_i) = 1/\lambda_i^2$;
2. $\mathbf{P}(X \geq X_1 + X_2 | X \geq X_1) = \mathbf{P}(X \geq X_1)$, so $\mathbf{P}(X \geq X_1 + X_2) = \mathbf{P}(X \geq X_1)\mathbf{P}(X \geq X_2)$;
3. Define $Y = min(X_i)$, then $Y \sim Exponential(\sum_i \lambda_i)$;
4. Define $Y = X_i - min_{j \neq i} X_j$, then $\mathbf{P}(Y < 0) = \frac{\lambda_i}{\sum \lambda_i}$;

8. Arrival Times for Poisson Processes If $N(t)$ is a Poisson process with rate λ, then the arrival times $T_n = \sum_{1,\cdots,n} X_i, \cdots$ have $Gamma(n, \lambda)$ distribution. Specifically,

$$E[T_n] = \frac{n}{\lambda}, \text{and } Var[T_n] = \frac{n}{\lambda^2}$$

The above discussion suggests a way to simulate (generate) a Poisson process's T_i.

9. Merging Independent Poisson Processes Let $N_{1,\cdots,m}(t)$ be m independent Poisson processes with rate $\lambda_{1,\cdots,m}$, let also

$$N(t) = \sum_{i=1,\cdots,m} N_i(t), \text{for all } t \in [0, \infty].$$

Then, $N(t)$ is a Poisson process with rate $\lambda \sim Poisson(\lambda \equiv \sum \lambda_i)$. .

10. Splitting a Poisson Processes Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ into two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands head up, the arrival is sent to the first process ($N_1(t)$). Otherwise, it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,

1. $N_1(t)$ is a Poisson process with rate λp;
2. $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$;
3. $N_1(t)$ and $N_2(t)$ are independent.

11. The Second Definition of the Poisson Process Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)]\}$ is called a Poisson process with **rate λ** if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent and stationary increments;

3. We have

$$P(N(\Delta) = 0) = 1 - \lambda\Delta + o(\Delta),$$
$$P(N(\Delta) = 1) = \lambda\Delta + o(\Delta),$$
$$P(N(\Delta) \leq 2) = o(\Delta).$$

12. Nonhomogeneous Poisson Process Let $\lambda(t) : [0, +\infty) \to [0, +\infty)$ be an integrable function. The counting process $\{N(t), t \in [0, \infty)]\}$ is called a **nonhomogeneous Poisson process** with rate with **rate** $\lambda(t)$ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. for any $t \in [0, +\infty)$, we have

$$P(N(t + \Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + o(\Delta),$$
$$P(N(t + \Delta) - N(t) = 1) = \lambda(t)\Delta + o(\Delta),$$
$$P(N(t + \Delta) - N(t) \leq 2) = o(\Delta).$$

Such a process has all the properties of a Poisson process except for the fact that its rate is a function of time, i.e., $\lambda = \lambda(t)$. For a nonhomogeneous Poisson process with rate $\lambda(t)$, the number of arrivals in any interval is a Poisson random variable; however, its parameter can depend on the location of the interval. More specifically, we can write

$$N(t + s) - N(t) \sim Poisson(\int_t^{t+s} \lambda(\alpha) d\alpha).$$